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On group rings of linear groups

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ABSTRACT

Let H be a finitely generated group of matrices over a field F of characteristic zero. We consider the group ring KH of H over an arbitrary field K whose characteristic is either zero or greater than some number $N = N(H)$. We prove that KH is isomorphic to a subring of a ring S which is a crossed product of a division ring Δ with a finite group. Hence KH is isomorphic to a subring of a matrix ring over a skew field.

1. Introduction

1.1. The main result of this paper is the following theorem.

Theorem 1. *Let H be a finitely generated subgroup of $GL_n(F)$, $char(F) = 0$, and let KH be the group ring of H over a field K.*

i) *If $char(K) = 0$ then there exists a torsion free normal subgroup G of finite index in H and an isomorphic embedding of KH into a semisimple Artinian ring S such that the group ring KG generates an H-invariant division subring $\Delta \subseteq S$ and S is isomorphic to a suitable cross product*

$$S \cong \Delta * (H/G) \tag{1.1}$$

where the isomorphism (1.1) extends the isomorphism $KH \cong KG * (H/G)$.

ii) *If $char(K)$ is finite and greater than N, a number depending on H, then statement* i) *remains true for KH.*

The additional information about the normal subgroup G and its group ring is contained in Theorems 2 and 3 below.

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Corollary 1. *Let H be a finitely generated subgroup of $GL_n(F)$, $char(F) = 0$, and let KH be the group ring of H over a field K with $char(K) = 0$ or $char(K) > N = N(H)$. Then there exists a division K-algebra Δ such that KH is isomorphic to a subring of a matrix ring $\Delta_{m\times m}$.*

The corollary follows from Theorem 1 immediately. In fact, S has a finite left dimension $m = (H : G)$ over Δ, hence it has a faithful representation in $\Delta_{m\times m}$.

It is worth remarking that the division ring Δ in the corollary will be commutative only if the group H has an abelian normal subgroup of finite index when $char(K) = 0$, or a p-abelian subgroup of finite index if $char(K) = p$. This follows from Corollaries 5.3.8–5.3.9 in Passman's book [12].

1.2. The proof of Theorem 1 is given in section 6. It is based on Theorems 2, 3 and 4.

We prove Theorem 2 in section 3; we obtain there a torsion free normal subgroup G of finite index in H such that the group ring of G over the ring of p-adic integers Ω has an H-invariant filtration with an associated graded ring isomorphic to the polynomial ring over a prime field Z_p. The proof of this theorem makes an essential use of Lazard's p-valuations (see Lazard [6]). A short description of Lazard's method is presented in subsection 2.2.

The second main step in the proof of Theorem 1 is Theorem 4 (section 5) whose proof is based on the method developed by the author in [9] and [10]; this method is described also in Cohn [3], section 2.6. We apply Theorem 2 and Theorem 4 to construct the division ring Δ in Theorem 1 in the case when $char(K) = 0$.

To prove statement ii) of Theorem 1 we need Theorem 3 and Corollary 3 (section 4) which states that the existence of a p-valuation in a group G implies that there exists a filtration and a valuation in the group ring Z_pG over the prime field Z_p with associated graded ring isomorphic to the polynomial ring over Z_p.

Our arguments show in fact that the conclusions of Theorems 1 and 2 remain true for an arbitrary, not necessarily finitely generated, subgroup $H \subseteq GL_n(T)$ if T is a finitely generated commutative domain of characteristic zero.

2. Preliminaries

2.1.

Lemma 1. *Let T be a finitely generated commutative domain of characteristic zero and of transcendence degree n, and $t_1, t_2, \cdots, t_n$ be a system of elements in T algebraically independent over Z. Then there exists a natural number N such that for every prime $p > N$ and natural number $c > N$ the powers of the ideal $A_{p,c}$ generated by the elements $p, t_1 - c, t_2 - c, \cdots, t_n - c$ define a p-adic valuation $\rho_{p,c}$ of T such that*

i) *$\rho_{p,c}(p) = 1$, and*
ii) *if $J(T) = \{r \in T | \rho_{p,c}(r) > 0\}$, then the quotient ring $T/J(T)$ is a finite field with characteristic p.*

Proof. Since T is finitely generated we can extend the system $t_1, t_2, \cdots, t_n$ to a system of elements $t_1, t_2, \cdots, t_n; s_1, s_2, \cdots, s_m$ which generates T. Let $\phi_j[x]$ $(1 \le j \le m)$ be the minimal polynomial of s_j over $Z[t_1, t_2, \cdots, t_n]$. We consider the field of fractions R of T and its subfield $S = Q(t_1, t_2, \cdots, t_n)$ which is the field of rational functions in variables $t_1, t_2, \cdots, t_n$ over the field Q of rational numbers. We pick an element θ such that $R = S(\theta)$ and let $\psi[x]$ be the minimal polynomial of θ; we can assume that all the coefficients of $\psi[x]$ belong to the subring $Z[t_1, t_2, \cdots, t_n]$ as well as its discriminant $d[t_1, t_2, \cdots, t_n]$.

We pick now an arbitrary prime number p and a natural number c and consider the ideal $A_{p,c} \subseteq Z[t_1, t_2, \cdots, t_n]$ generated by the system of elements $p, t_1 - c, t_2 - c, \cdots, t_n - c$. The quotient ring