Journal of Pure and Applied Algebra 221 (2017) 36–44

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

On a theorem by Brewer

Le Thi Ngoc Giau, Byung Gyun Kang *

Department of Mathematics, Pohang University of Science and Technology, Pohang 37673, Republic of Korea

ARTICLE INFO

Article history:
Received 21 October 2015
Available online 2 June 2016
Communicated by A.V. Geramita

MSC:
13A15; 13C15; 13F25

ABSTRACT

One of the most frequently referenced monographs on power series rings, "Power Series over Commutative Rings" by James W. Brewer, states in Theorem 21 that if M is a non-SFT maximal ideal of a commutative ring R with identity, then there exists an infinite ascending chain of prime ideals in the power series ring $R[\![X]\!]$, $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n \subsetneq \cdots$ such that $Q_n \cap R = M$ for each n. Moreover, the height of $M[\![X]\!]$ is infinite. In this paper, we show that the above theorem is false by presenting two counter examples. The first counter example shows that the height of $M[\![X]\!]$ can be zero (and hence there is no chain $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n \subsetneq \cdots$ of prime ideals in $R[\![X]\!]$ satisfying $Q_n \cap R = M$ for each n). In this example, the ring R is one-dimensional. In the second counter example, we prove that even if the height of $M[\![X]\!]$ is uncountably infinite, there may be no infinite chain $\{Q_n\}$ of prime ideals in $R[\![X]\!]$ satisfying $Q_n \cap R = M$ for each n.

1. Introduction

In this paper, a ring R always means a commutative ring with identity. The Krull dimension of R is denoted by $\dim R$. If P is a prime ideal of R, then the height of P, denoted by ht$P$, is defined by $\dim R_P$. An ideal I of R is called an *SFT ideal* if there exist a finitely generated ideal $J \subseteq I$ and a positive integer k such that $a^k \in J$ for each $a \in I$. A ring R is called an *SFT ring* if every ideal of R is an SFT ideal.

Seidenberg showed in [12] that if the Krull dimension of R is finite, then so is that of the polynomial ring $R[X]$. In fact, he proved the bound: $\dim R + 1 \le \dim R[X] \le 2\dim R + 1$. He also showed that any intermediate values can be obtained by appropriately choosing R [13]. Therefore, the possibilities for the Krull dimension of the polynomial ring $R[X]$ are completely determined.

For the power series ring $R[\![X]\!]$, it is easy to show that $\dim R + 1 \le \dim R[\![X]\!]$. However, it is not always true that $\dim R[\![X]\!] \le 2\dim R + 1$ even if $\dim R[\![X]\!] < \infty$. A counter example was given by Kang and Park in [8]. In general, it may happen that $\dim R[\![X]\!] = \infty$ even if $\dim R$ is finite [1] (see also [3,6,7,9–11]). In [1],

* Corresponding author.
E-mail addresses: ngocgiau@postech.ac.kr (L.T.N. Giau), bgkang@postech.ac.kr (B.G. Kang).

http://dx.doi.org/10.1016/j.jpaa.2016.05.018

Arnold showed that $\dim R[\![X]\!] = \infty$ when R is a non-SFT ring. In fact, he constructed an infinite ascending chain of prime ideals in $R[\![X]\!]$, $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n \subsetneq \cdots$, when R is a non-SFT ring. One of the most frequently referenced monographs on power series rings, "Power Series over Commutative Rings" by James W. Brewer [2], states in Theorem 21 that if M is a non-SFT maximal ideal of a ring R, then there exists an infinite ascending chain of prime ideals in the power series ring $R[\![X]\!]$, $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n \subsetneq \cdots$ such that $Q_n \cap R = M$ for each n. Moreover, the height of $M[\![X]\!]$ is infinite. However, in the 1982 review of [2] in Mathematical Reviews [5], Gilmer noticed that the proof of the theorem is incomplete and the status of the result is uncertain. Ever since, the status of the theorem has remained open. In this paper, we show that this theorem is false by presenting two counter examples. In the first counter example (Section 2), we show that there exists a one-dimensional ring R with a non-SFT maximal ideal $\overline{M}$ such that $\mathrm{ht}\overline{M}[\![X]\!] = 0$, i.e., $\overline{M}[\![X]\!]$ is a minimal prime ideal, and such that if Q is a prime ideal of $R[\![X]\!]$ satisfying $Q \cap R = \overline{M}$, then either $Q = \overline{M}[\![X]\!]$ or $Q = \overline{M} + (X)$. In particular, there is no infinite chain $\{Q_j\}$ of prime ideals in $R[\![X]\!]$ such that $Q_j \cap R = \overline{M}$ for each j. In the second counter example (Section 3), we show that there exists an (infinite-dimensional) valuation domain V with non-SFT maximal ideal M such that $\mathrm{ht}M[\![X]\!]$ is uncountably infinite and such that if Q is a prime ideal of $V[\![X]\!]$ satisfying $Q \cap V = M$, then either $Q = M[\![X]\!]$ or $Q = M + (X)$. In particular, there is no infinite chain $\{Q_j\}$ of prime ideals in $V[\![X]\!]$ such that $Q_j \cap V = M$ for each j. In other words, there does exist an infinite chain of prime ideals in $V[\![X]\!]$ lying inside $M[\![X]\!]$. However, none of the prime ideals in the chain other than $M[\![X]\!]$ has contraction M in V. In each of the two counter examples, we also describe the spectrum of the corresponding ring, determine the SFT property of P and calculate (or give the possibilities for) height of $P[\![X]\!]$ for each prime ideal P of the ring.

2. First counter example

In this section, we show that there is a ring R with a non-SFT maximal ideal $\overline{M}$ such that $\mathrm{ht}\overline{M}[\![X]\!] = 0$ and such that if Q is a prime ideal of $R[\![X]\!]$ satisfying $Q \cap R = \overline{M}$, then either $Q = \overline{M}[\![X]\!]$ or $Q = \overline{M} + (X)$. In particular, there is no infinite chain $\{Q_j\}$ of prime ideals in $R[\![X]\!]$ such that $Q_j \cap R = \overline{M}$ for each j.

We now construct the ring R. Let k be a field and let Ω be the first uncountable ordinal number. Let $\mathcal{X} = \{x_\alpha\}_{\alpha \in \Omega}$ be a set of indeterminates over k. For $x_\alpha, x_\beta \in \mathcal{X}$, we say $x_\alpha > x_\beta$ if $\alpha > \beta$ in Ω. Let

$$R = k[\mathcal{X}]/I,$$

where

$$I = (\{(u+1)v \mid u, v \in \mathcal{X}, u > v\})$$

is the ideal of $k[\mathcal{X}]$ generated by all polynomials $(u+1)v$ with $u, v \in \mathcal{X}$ such that $u > v$. Let

$$M = (\{u \mid u \in \mathcal{X}\})$$

be the ideal of $k[\mathcal{X}]$ generated by all u with $u \in \mathcal{X}$. For an ideal J of $k[\mathcal{X}]$, we denote by $\overline{J}$ the image of J in R (through the natural projection $k[\mathcal{X}] \to k[\mathcal{X}]/I = R$). Also, for each $f \in k[\mathcal{X}]$, we denote by $\overline{f}$ the image of f in R.

For each $u \in \mathcal{X}$, let $\varphi_u : k[\mathcal{X}] \to k[u]$ be the map induced by

$$v \mapsto \begin{cases} 0 & \text{if } v < u \\ -1 & \text{if } v > u \\ u & \text{if } v = u \end{cases}.$$