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AF inverse monoids and the structure of countable MV-algebras ☆

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ABSTRACT

This paper is a further contribution to the developing theory of Boolean inverse monoids. These monoids should be regarded as non-commutative generalizations of Boolean algebras; indeed, classical Stone duality can be generalized to this non-commutative setting to yield a duality between Boolean inverse monoids and a class of étale topological groupoids. MV-algebras are also generalizations of Boolean algebras which arise from many-valued logics. It is the goal of this paper to show how these two generalizations are connected. To do this, we define a special class of Boolean inverse monoids having the property that their lattices of principal ideals naturally form an MV-algebra. We say that an arbitrary MV-algebra can be co-ordinatized if it is isomorphic to an MV-algebra arising in this way. Our main theorem is that every countable MV-algebra can be so co-ordinatized. The particular Boolean inverse monoids needed to establish this result are examples of what we term AF inverse monoids and are the inverse monoid analogues of AF C^*-algebras. In particular, they are constructed from Bratteli diagrams as direct limits of finite direct products of finite symmetric inverse monoids.

1. Introduction

MV-algebras were introduced by C.C. Chang in 1958 [13] in his algebraic studies of many valued logics. In Chang's original axiomatization, it is plain that such algebras are generalizations of Boolean algebras. In general, the elements of an MV-algebra are not idempotent, but those that are form a Boolean algebra.

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A good introduction to their theory may be found in Mundici's tutorial notes [46]. The standard reference is [14]. The starting point for our paper is Mundici's own work that connects countable MV-algebras to a class of AF C^*-algebras [43,47]. He sets up a correspondence between AF C^*-algebras whose Murray–von Neumann order is a lattice and countable MV-algebras. In [45], he argues that AF algebras 'should be regarded as sort of noncommutative Boolean algebras'. This is persuasive because the commutative AF C^*-algebras are function algebras over separable Boolean spaces. But the qualification 'sort of' is important. The result would be more convincing if commutative meant, precisely, countable Boolean algebra. In this paper, we shall introduce a class of countable structures whose commutative members are precisely this.

Approximately finite (AF) C^*-algebras, that is those C^*-algebras which are direct limits of finite dimensional C^*-algebras, were introduced by Bratteli in 1972 [10], and form one of the most important classes of C^*-algebras. Reading Bratteli's paper, it quickly becomes apparent that his calculations rest significantly on the properties of matrix units. The reader will recall that these are square matrices all of whose entries are zero except for one place where the entry is one. Our key observation is that matrix units of a given size n form a groupoid, and this groupoid determines the structure of a finite symmetric inverse monoid on n letters. The connection is via what are termed rook matrices [54]. Symmetric inverse monoids are simply the monoids of all partial bijections of a given set. Here the set can be taken to be $\{1, \ldots, n\}$. These monoids have a strong Boolean character. For example, their semilattices of idempotents form a finite Boolean algebra. They are however non-commutative. This leads us to define a general class of Boolean inverse monoids, called AF inverse monoids, constructed from Bratteli diagrams. We argue that this class of monoids is the most direct non-commutative generalization of Boolean algebras. For example, they figure in the developing theory of non-commutative Stone dualities [36–40] where they are associated with a class of étale topological groupoids. Significantly, commutative AF inverse monoids are countable Boolean algebras. It is worth noting that the groups of units of such inverse monoids have already been studied [15,28,31] but without reference to inverse monoids.

We prove that the poset of principal ideals of an AF inverse monoid naturally forms an MV-algebra when that poset is a lattice. Accordingly, we say that an MV-algebra that is isomorphic to an MV-algebra constructed in this way may be co-ordinatized by an inverse monoid. The main theorem we prove in this paper is that *every* countable MV-algebra may be co-ordinatized in this way. As a concrete example, we provide an explicit description of the AF inverse monoid that co-ordinatizes the MV-algebra of dyadic rationals in the unit interval. It turns out to be a discrete version of the CAR algebra. Finally, our results also can be viewed as contributing to the study of the poset of $\mathscr{J}$-classes of an inverse semigroup. For results in this area and further references, see [41]. There are also *thematic* links between our work and that to be found in [4,22,55]. This has influenced our choice of terminology when referring to partial refinement monoids. Such partial monoids, including so-called effect algebras, are currently an active research area [23] and provide a useful general framework for our coordinatization theorem.

Since our paper appeared in the arXiv,[1] Friedrich Wehrung [56] has developed some of its ideas. In particular, he has proved that every MV-algebra can be co-ordinatized by a Boolean inverse monoid using different methods.

2. Basic definitions

We shall work with two classes of structures in this paper: inverse monoids and partial commutative monoids. The goal of this section is to define the structures we shall be working with, and state precisely what we mean by co-ordinatizing an MV-algebra by means of an inverse monoid.

[1] arXiv:1408.1231v2.