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Projective varieties of maximal sectional regularity

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A B S T R A C T

We study projective varieties $X \subset \mathbb{P}^r$ of dimension $n \geq 2$, of codimension $c \geq 3$ and of degree $d \geq c+3$ that are of maximal sectional regularity, i.e. varieties for which the Castelnuovo–Mumford regularity $\operatorname{reg}(\mathcal{C})$ of a general linear curve section is equal to $d-c+1$, the maximal possible value (see [10]). As one of the main results we classify all varieties of maximal sectional regularity. If X is a variety of maximal sectional regularity, then either (a) it is a divisor on a rational normal $(n+1)$-fold scroll $Y \subset \mathbb{P}^{n+3}$ or else (b) there is an n-dimensional linear subspace $\mathbb{F} \subset \mathbb{P}^r$ such that $X \cap \mathbb{F} \subset \mathbb{F}$ is a hypersurface of degree $d-c+1$. Moreover, suppose that $n = 2$ or the characteristic of the ground field is zero. Then in case (b) we obtain a precise description of X as a birational linear projection of a rational normal n-fold scroll.

1. Introduction

Let $X \subset \mathbb{P}^r$ be a nondegenerate irreducible projective variety of dimension n, codimension $c > 1$ and degree d over an algebraically closed field $\Bbbk$. D. Mumford [15] has defined X to be *m-regular* if its ideal sheaf $\mathcal{I}_X$ satisfies the following vanishing condition

$$H^i(\mathbb{P}^r, \mathcal{I}_X(m-i)) = 0 \text{ for all } i \geq 1.$$

Mumford's homological description grows out of Castelnuovo's classical results (see [5]). The m-regularity condition implies the $(m+1)$-regularity condition, so that one defines the *Castelnuovo–Mumford regularity* $\operatorname{reg}(X)$ of X as the least integer m such that X is m-regular. It is well known that if X is m-regular then its homogeneous ideal is generated by forms of degree $\leq m$. This algebraic implication of m-regularity has

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an elementary geometric consequence that any $(m+1)$-secant line to X should be contained in X. We say that a linear space $L \subset \mathbb{P}^r$ is k-secant to X if

$$\mathrm{dim(X \cap L) = 0 \quad and \quad length(X \cap L) := dim_{\Bbbk}(\mathcal{O}_{\mathbb{P}^r}/\mathcal{I}_X + \mathcal{I}_L) \geq k.}$$

A well known conjecture due to Eisenbud and Goto (see [6]) says that

$$\mathrm{reg}(X) \leq d - c + 1. \tag{1.1}$$

Obviously this conjecture implies the following conjecture

$$X \text{ has no proper } k\text{-secant line if } k > d - c + 1. \tag{1.2}$$

So far the conjecture (1.1) has been proved only for irreducible but not necessarily smooth curves by Gruson–Lazarsfeld–Peskine [10] and for smooth complex surfaces by H. Pinkham [20] and R. Lazarsfeld [14]. Moreover, in [10] the curves in $\mathbb{P}^r$ whose regularity takes the maximally possible value $d-r+2$ are completely classified: they are either of degree $\leq r+1$ or else smooth rational curves having a $(d-r+2)$-secant line. The statement (1.2) is known to be true when X is locally Cohen–Macaulay (see Theorem 1 in [17]). But it is still unknown for arbitrary varieties.

The main subject of the present paper is to study the geometry of proper $(d-c+1)$-secant lines to a projective variety. To this aim, we investigate the *extremal secant locus* $\Sigma(X)$ of X, that is, the closure of the set of all proper $(d-c+1)$-secant lines to X in the Grassmannian $\mathbb{G}(1, \mathbb{P}^r)$. Of course, if the extremal secant locus of X is nonempty then its regularity is at least $d-c+1$ and so such a variety will play an important role in the natural problem of classifying all extremal varieties with respect to the above regularity conjecture. For $d \geq c+3$, Gruson–Lazarsfeld–Peskine's result in [10] provides a complete classification of curves having a $(d-c+1)$-secant line. They should be smooth and rational. M.A. Bertin [1] generalizes this result to higher dimensional smooth varieties. She proves the conjecture (1.1) for smooth rational scrolls – which is reproved in [13] – and shows that if X is a smooth variety having a $(d-c+1)$-secant line then it should be the linear regular projection of a smooth rational normal scroll. Later, A. Noma [17] obtains a very nice description of those smooth rational scrolls.

In Theorem 3.4 we show that if $c \geq 3$ and $d \geq c+3$, then the dimension of $\Sigma(X)$ is at most $2n-2$ and the equality is attained if and only if a general linear curve section of X has the maximal Castelnuovo–Mumford regularity $d-c+1$. We will say that X is a *variety of maximal sectional regularity* if its general linear curve section is of maximal regularity (cf. [4]).

To complete the result starting with Theorem 3.4, it is natural to ask for a classification of all varieties of maximal sectional regularity. This is the contents of Theorem 6.3 and Theorem 7.1. More precisely, for $c \geq 3$ and $d \geq c+3$ we obtain a classification of surfaces of maximal sectional regularity in Theorem 6.3 and a classification of higher dimensional varieties of maximal sectional regularity in Theorem 7.1. It turns out that $X \subset \mathbb{P}^r$ is variety of maximal sectional regularity if and only if it is one of the followings:

(a) $c = 3$ and X is a divisor of the $(n+1)$-fold scroll $Y = S(\underbrace{0, \ldots, 0}_{(n-2)\text{-times}}, 1, 1, 1) \subset \mathbb{P}^{n+3}$ such that X is linearly equivalent to $H + (d-3)F$, where H is the hyperplane divisor of Y and $F \subset Y$ is a linear subspace of dimension n;
(b) There exists an n-dimensional linear subspace $\mathbb{F} \subset \mathbb{P}^r$ such that $X \cap \mathbb{F}$ in $\mathbb{F}$ is a hypersurface of degree $d-c+1$.

In particular, there exist varieties $X \subset \mathbb{P}^r$ of maximal sectional regularity of dimension n, of codimension c and of degree d for any given (n, c, d) with $n \geq 2$, $c \geq 3$ and $d \geq c+3$. Furthermore, assume that $\mathrm{char}(\Bbbk) = 0$