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Interacting Hopf algebras

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ABSTRACT

We introduce the theory $\mathbb{IH}_{\mathsf{R}}$ of interacting Hopf algebras, parametrised over a principal ideal domain R. The axioms of $\mathbb{IH}_{\mathsf{R}}$ are derived using Lack's approach to composing PROPs: they feature two Hopf algebra and two Frobenius algebra structures on four different monoid–comonoid pairs. This construction is instrumental in showing that $\mathbb{IH}_{\mathsf{R}}$ is isomorphic to the PROP of linear relations (i.e. subspaces) over the field of fractions of R.

1. Introduction

We introduce the theory of Interacting Hopf Algebras, characterising linear relations. Its equations are obtained via Lack's composition of PROPs [24].

Diagrammatic formalisms are widespread in various fields, including computer science, control theory, logic and quantum information [4]. Several recent approaches [1,14,33,20,28,18,7,9,3,11,6] consider diagrams rigorously as the arrows of a *symmetric monoidal theory* (SMTs). By SMT we mean a presentation of a PROP: a set of generators—the *syntax* of diagrams—together with a set of *equations* that, in conjunction with the usual laws of symmetric monoidal categories, give the notion of diagram equality. Of particular importance are SMT featuring both algebraic and coalgebraic structure, subject to compatibility conditions: notable examples are Frobenius algebras and bialgebras whose equations witness an *interaction* between a commutative monoid and cocommutative comonoid.

Lack [24] showed that several such situations can be understood as arising from *PROP composition* where a *distributive law*—a notion closely related to standard distributive laws between monads [34]—witnesses the interaction. The beauty of this approach is that one can consider distributive laws to be responsible for the newly introduced equations, resulting in a pleasantly modular account of the composite algebraic theory. For example, the equations of (strongly separable) Frobenius algebra [12] can be obtained in this

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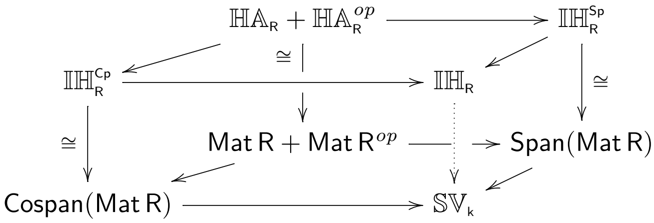
way. Another example is the theory of bialgebras: here monoids and comonoids interact through a different distributive law, thus yielding different equations.

Our chief original contribution is the study of the interaction of the PROP $\mathbb{HA}_{\mathsf{R}}$ of Hopf algebras, parametrised over a principal ideal domain R, and its opposite $\mathbb{HA}_{\mathsf{R}}^{op}$. As in the case of the PROP of commutative monoids and its opposite, two different distributive laws can be defined, yielding $\mathbb{IH}_{\mathsf{R}}^{\mathsf{Sp}}$ and $\mathbb{IH}_{\mathsf{R}}^{\mathsf{Cp}}$ respectively. Our main theory of interest $\mathbb{IH}_{\mathsf{R}}$ is the result of merging together these two equational theories. These ingredients constitute the topmost face in the following commutative cube in the category of PROPs.



(⧉)

The bottom face of (⧉) describes the linear algebraic nature of our SMTs. First, $\mathbb{HA}_{\mathsf{R}}$ is isomorphic to the PROP $\mathsf{Mat}\,\mathsf{R}$ of R-matrices. Second, since the equations of $\mathbb{IH}_{\mathsf{R}}^{\mathsf{Sp}}$ and $\mathbb{IH}_{\mathsf{R}}^{\mathsf{Cp}}$ arise from distributive laws, these SMTs isomorphic to PROPs of spans and cospans of R-matrices, respectively—these latter PROPs exist because $\mathsf{Mat}\,\mathsf{R}$ has pullbacks and pushouts whenever R is a principal ideal domain. The isomorphism between $\mathbb{IH}_{\mathsf{R}}$ and $\mathbb{SV}_{\mathsf{k}}$ follows from the fact that the top and the bottom faces of (⧉) are pushouts. $\mathbb{SV}_{\mathsf{k}}$ is the PROP of linear relations over the field k of R-fractions: an arrow $n \to m$ is a k-linear subspace of $\mathsf{k}^n \times \mathsf{k}^m$, composition is relational.

We contend that $\mathbb{IH}_{\mathsf{R}}$ is a *canonical syntax for (finite dimensional) linear algebra*: linear transformations, spaces, kernels, etc. are all represented faithfully in the graphical language. This perspective will be pursued in the paper: several proofs mimic—at the diagrammatic level—familiar techniques such as Gaussian elimination. We believe that the string-diagrammatic treatment of linear algebra is of cross-disciplinary benefit: indeed, some applications of the theory herein have already been developed; see below.

Applications and related work. For different choices of R, the theory of interacting Hopf algebras has several applications in diverse disciplines. A particularly interesting instance is the polynomial ring $\mathsf{R} = \mathbb{R}[x]$: $\mathbb{IH}_{\mathbb{R}[x]}$ is a string-diagrammatic account of *signal-flow graphs*, which are foundational structures of control theory and signal processing that capture behaviour defined via recurrence relations/differential equations. $\mathbb{IH}_{\mathbb{R}[x]}$ provides a formal syntax and semantics, a sound and complete equational theory and an analogue of Kleene's theorem [23] stating that all rational behaviours can be denoted within $\mathbb{IH}_{\mathbb{R}[x]}$. The interested reader is referred to [7,9,19] for a closer look to this ramification of our work and to [31,27] for other categorical models of signal flow graphs.

After the submission of [7] and the appearance of an earlier version of this manuscript on arXiv (http://arxiv.org/abs/1403.7048), Baez and Erbele [3] independently gave an equivalent presentation of $\mathbb{IH}_{\mathbb{R}[x]}$. The main difference is our use of distributive laws, which enables us to obtain $\mathbb{IH}_{\mathsf{R}} \cong \mathbb{SV}_{\mathsf{k}}$ using universal properties as well as the span/cospan factorisations in $\mathbb{IH}_{\mathsf{R}}$.

An earlier conference version of this work appeared in [8] and only considered the theory $\mathbb{IH}_{\mathbb{Z}_2}$, which also has significant applications. First, it is closely related to the algebra of stateless connectors [10], modelling concurrent interactions of software components. Second, it is the phase-free fragment of the ZX-calculus [14], an SMT for interacting quantum observables which originated in the research programme of categorical quantum mechanics [1]. Completeness for ZX has been intensively studied in recent years [29,16,2] and our work yields a free model $\mathbb{SV}_{\mathbb{Z}_2}$ for the phase-free fragment. Our modular analysis also gives new insights about the algebra of quantum theories: while the Frobenius structures have traditionally been regarded as