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On the tensor product of modules over skew monoidal actegories

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ABSTRACT

This paper is about skew monoidal tensored $\mathcal{V}$-categories (= skew monoidal hommed $\mathcal{V}$-actegories) and their categories of modules. A module over $\langle \mathcal{M}, *, R\rangle$ is an algebra for the monad $T = R * \_$ on $\mathcal{M}$. We study in detail the skew monoidal structure of $\mathcal{M}^T$ and construct a skew monoidal forgetful functor $\mathcal{M}^T \to {}_E\mathcal{M}$ to the category of E-objects in $\mathcal{M}$ where $E = \mathcal{M}(R, R)$ is the endomorphism monoid of the unit object R. Then we give conditions for the forgetful functor to be strong monoidal and for the category $\mathcal{M}^T$ of modules to be monoidal. In formulating these conditions a notion of 'self-cocomplete' subcategories of presheaves appears to be useful which provides also some insight into the problem of monoidality of the skew monoidal structures found by Altenkirch, Chapman and Uustalu on functor categories $[\mathcal{C}, \mathcal{M}]$.

1. Introduction

A skew monoidal category consists of a category $\mathcal{M}$, a functor $\mathcal{M} \times \mathcal{M} \xrightarrow{*} \mathcal{M}$, an object $R \in \mathcal{M}$ and comparison morphisms $\gamma : L * (M * N) \to (L * M) * N$, $\eta : M \to R * M$, $\varepsilon : M * R \to M$ satisfying usual monoidal category axioms but without the assumption that they are isomorphisms. This structure, more primitive than a monoidal category, turned out to contain all algebraic information about a bialgebroid over a non-commutative ring R. As it was shown in [19, Theorem 9.1] the closed skew monoidal structures on **Mod**-R with unit object being the right regular R-module are precisely the right bialgebroids B over R. Under this correspondence the category of right B-modules becomes, not a 'module category' over the skew monoidal $\mathcal{M}$ but simply, the Eilenberg–Moore category $\mathcal{M}^T$ of the monad $T = R * \_$ on $\mathcal{M}$. This canonical monad is present also in any monoidal category although it is an uninteresting identity monad. For skew monoidal categories, however, it is the structure of $\mathcal{M}^T$ that embodies the representation theory of the skew monoidal $\mathcal{M}$ if viewed as a generalized bialgebroid.

It is well-known that the category of modules over a bialgebra, weak bialgebra or bialgebroid B has a monoidal structure. It is defined using the coalgebra structure of B and the forgetful functor to the

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underlying monoidal category **Mod**-k or R-**Mod**-R. In case of a skew monoidal $\mathcal{M}$ there is no a priori given underlying monoidal category, only the category $\mathcal{M}$ itself. This makes the construction of any (skew?) monoidal product on $\mathcal{M}^T$ non-trivial and in general impossible unless certain right exactness conditions are fulfilled. In the first version of this paper (presented in a talk [20]) we have constructed such a skew tensor product provided $\mathcal{M}$ had reflexive coequalizers and $*$ preserved them. Independently, S. Lack and R. Street have given in [14] the same definition of the tensor but noticed also that it exists if $*$ preserves reflexive coequalizers only in the second argument. We call the tensor product $\bar{\otimes}$ on $\mathcal{M}^T$ the 'horizontal tensor' in the hope that the 'vertical tensor' of comodules will also be found, although the latter so far has resisted all attempts.

Having defined the skew monoidal structure $\langle \mathcal{M}^T, \bar{\otimes}, \bar{R}\rangle$ on the modules over $\mathcal{M}$ the next step is to construct a skew monoidal forgetful functor to some underlying skew monoidal category. The natural candidate for the underlying category is the category ${}_E\mathcal{M}$ of E-objects in $\mathcal{M}$ where E denotes the endomorphism monoid of R. In case of an R-bialgebroid ${}_E\mathcal{M}$ is just the bimodule category R-**Mod**-R. The skew monoidal product $\underline{\otimes}$ on ${}_E\mathcal{M}$ can be introduced in two equivalent ways: Either as the horizontal tensor product of an even more primitive skew monoidal structure on $\mathcal{M}$ (which uses only the object R but not $*$), or as the Altenkirch–Chapman–Uustalu construction [3] applied to ${}_E\mathcal{M} \equiv [E, \mathcal{M}]$ by considering E as a 1-object category. The justification for this choice of $\underline{\otimes}$ is that a monadic and skew monoidal functor $\mathcal{G} : \mathcal{M}^T \to {}_E\mathcal{M}$ can be constructed.

We do all these constructions in the framework of tensored $\mathcal{V}$-categories. Assuming the existence of tensors $V \odot M$ of $V \in \mathcal{V}$ and $M \in \mathcal{M}$ is a technical assumption which, by the equivalence [9,10] between enriched categories [12] with tensors and actegories [17] with hom objects, allows us to work with ordinary functors and natural transformations. This does not mean that skew monoidal categories without tensors are uninteresting. Skew monoidal monoids [21] are good examples. Only in Section 8 we make an exception by studying general $\mathcal{V}$-categories $\mathcal{M}$ in order to characterize the functor categories $[\mathcal{C}, \mathcal{M}]$ for which the skew monoidal structure of [3] is monoidal. Introducing *self-cocompleteness* we find that if $\mathcal{M}$ is a self-cocomplete subcategory of presheaves over $\mathcal{C}$, or, in the case of $\mathcal{C} = E$, it is a self-cocomplete subcategory $\mathcal{W} \subseteq \mathcal{V}_E$ of right E-modules then $[\mathcal{C}, \mathcal{M}]$, respectively ${}_E\mathcal{M}$, is a monoidal category. Under appropriate right exactness conditions on $\langle \mathcal{M}, *, R\rangle$ this implies that $\langle \mathcal{M}^T, \bar{\otimes}, \bar{R}\rangle$ is also monoidal and the forgetful functor $\mathcal{G}$ is strong monoidal.

2. Preparations

2.1. Actegories

Let $\langle \mathcal{V}, \odot, I, a, l, r\rangle$ be a monoidal category in which the orientation of the coherence isomorphisms

$$a_{U,V,W} : U \odot (V \odot W) \xrightarrow{\sim} (U \odot V) \odot W$$

$$l_V : V \xrightarrow{\sim} I \odot V$$

$$r_V : V \odot I \xrightarrow{\sim} V$$

are chosen according to the right skew monoidal convention [19]. Therefore the monoidal category axioms (the redundant set of 5 axioms [16]), when written without inverses of a, l or r, look precisely as the skew monoidal[1] category axioms.

[1] Throughout this paper we use the term 'skew monoidal' for 'right skew monoidal'.