



Abelian subgroup separability of certain generalized free products of free or finitely generated nilpotent groups



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ABSTRACT

In this paper we prove that certain generalized free products of abelian subgroup separable groups, amalgamating an infinite cyclic subgroup, are abelian subgroup separable. Applying this, we derive that tree products of free groups or finitely generated nilpotent groups, amalgamating infinite cyclic subgroups, are abelian subgroup separable.

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1. Introduction

Let S be a subset of a group G . Then G is said to be S -separable if, for each $x \in G \setminus S$, there exists a normal subgroup N of finite index in G (briefly, $N \triangleleft_f G$) such that $x \notin NS$. If G is $\{1\}$ -separable, then G is said to be *residually finite*. If G is $\langle x \rangle$ -separable for each $x \in G$, then G is called *cyclic subgroup separable*. If G is H -separable for each finitely generated subgroup H of G , then G is called *subgroup separable* or *LERF* (locally extended residually finite). If G is H -separable for each abelian subgroup H of G , then G is called *abelian subgroup separable*. Here we allow that the abelian subgroup H may not be finitely generated.

Separability properties are interesting in both group theory and topology (see [7,17]). It is well-known that the word problem is solvable for finitely presented residually finite groups. In the same way, if a finitely presented group G is subgroup separable, then we can solve the generalized word problem for G . In the context of topology, subgroup separability can be used to solve immersion to embedding problems. For example, in 3-manifold topology, subgroup separability allows the passage from immersed incompressible surfaces to embedded incompressible surfaces in finite covers [9]. It therefore makes sense (especially in light

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of the fact that there are closed 3-manifolds M for which $\pi_1(M)$ is not subgroup separable [4]) to ask for separability only for some mildly restricted class of subgroups. In this note, we consider the groups which are separable for each abelian subgroup.

Free groups [8] and finitely generated nilpotent groups [13] are subgroup separable, hence they are cyclic subgroup separable and residually finite. Note that abelian subgroups of free groups are cyclic and subgroups of finitely generated nilpotent groups are finitely generated. Hence free groups and finitely generated nilpotent groups are abelian subgroup separable.

Generalized free products of two free groups, amalgamating a cyclic subgroup, are subgroup separable [3]. This result is one of the typical results in the study of the subgroup separability of generalized free products. Using the method in [3], the subgroup separability of generalized free products of subgroup separable groups was considered in [2,15]. In general, generalized free products of subgroup separable groups are not subgroup separable. For example, generalized free products of finitely generated nilpotent groups, amalgamating a cyclic subgroup, may not be subgroup separable [1]. However, generalized free products of finitely generated nilpotent groups, amalgamating a cyclic subgroup, are cyclic subgroup separable [6]. So we are interested in knowing if such generalized free products are abelian subgroup separable or not. In this direction, Hamilton [9] showed free products of abelian subgroup separable groups are abelian subgroup separable and fundamental groups of Haken 3-manifolds are abelian subgroup separable.

In this paper we are interested in the groups G satisfying the following:

(C1) If $H \leq G$ and $x \sim_G y$ for $x, y \in H$ then $x = y$.

For a free group or a finitely generated nilpotent group G , (C1) holds for an infinite cyclic subgroup H [5]. Clearly, if H is a central subgroup of G , then (C1) holds. We give a criterion that certain generalized free products of abelian subgroup separable groups satisfying (C1), amalgamating an infinite cyclic subgroup, are abelian subgroup separable (Theorem 3.6). Applying this, it is quite natural to derive that tree products of free groups or finitely generated nilpotent groups, amalgamating infinite cyclic subgroups, are abelian subgroup separable (Corollary 3.9).

Throughout this paper we use standard notation and terminology.

For $g \in G$, x^g and H^g denote $g^{-1}xg$ and $g^{-1}Hg$, respectively.

$x \sim_G y$ means that x and y are conjugate in G , otherwise $x \not\sim_G y$.

$N \triangleleft_f G$ denotes that N is a normal subgroup of finite index in G .

If A, B are groups, then $A *_H B$ denotes the generalized free product of A and B amalgamating the subgroup H .

If $x \in G = A *_H B$ then $\|x\|$ denotes the free product length of x in G .

Since a free group is subgroup separable [8], its finite extension is subgroup separable. Hence we have the following result.

Theorem 1.1. *Free-by-finite groups are subgroup separable. Hence if A and B are finite then $A *_H B$ is subgroup separable.*

2. Abelian subgroups of generalized free products

In order to study the abelian subgroups of generalized free products, we need to know the structure of their subgroups, which was given by Karrass and Solitar in [10].

Theorem 2.1. [10, Theorem 5 and Corollary] *Let $G = A *_U B$ and let H be a subgroup of G . Then $H = N \rtimes F$, where N is the tree product of $C_g = A^g \cap H$ or $B^g \cap H$ for some $g \in G$ and F is a free group.*

Now we can give information of abelian subgroups in generalized free products.

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