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Injectivity and surjectivity of the Dress map

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A R T I C L E I N F O

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A B S T R A C T

For a nontrivial finite Galois extension L/k (where the characteristic of k is different from 2) with Galois group G, we prove that the Dress map $h_{L/k} : A(G) \to GW(k)$ is injective if and only if $L = k(\sqrt{\alpha})$ where α is not a sum of squares in $k^\times$. Furthermore, we prove that $h_{L/k}$ is surjective if and only if k is quadratically closed in L. As a consequence, we give strong necessary conditions for faithfulness of the Heller–Ormsby functor $c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k$, as well as strong necessary conditions for fullness of $c^*_{L/k}$.

1. Introduction

Let k be a field of characteristic different from 2 and let L/k be a finite Galois extension with Galois group G. Let $A(G)$ denote the Burnside ring of G and let $GW(k)$ denote the Grothendieck–Witt ring of k. Recall that, as abelian groups, $A(G)$ is freely generated under disjoint union by cosets G/H where H runs through a set of representatives for conjugacy classes of subgroups, and $GW(k)$ is generated by 1-dimensional quadratic forms $\langle a\rangle$, where a runs through the group of square classes $k^\times/(k^\times)^2$, under orthogonal sum $\langle a\rangle + \langle b\rangle = \langle a, b\rangle$. Multiplication in $A(G)$ is given by cartesian product with identity G/G and multiplication in $GW(k)$ is given by the Kronecker product $\langle a\rangle\langle b\rangle = \langle ab\rangle$ with identity $\langle 1\rangle$. Following the construction in [1, Appendix B], the Dress map $h_{L/k} : A(G) \to GW(k)$ is a ring homomorphism that takes the coset G/H to the trace form $\mathrm{tr}_{L^H/k}\langle 1\rangle_{L^H}$, the quadratic form $x \mapsto \mathrm{tr}_{L^H/k}x^2$. (Our restriction on the characteristic of k is necessary for $h_{L/k}$ to be well defined.)

A particular point of interest is that the Dress map appears naturally in the study of equivariant and motivic stable homotopy theory. Heller and Ormsby [2, §4] construct a strong symmetric monoidal triangulated functor $c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k$ from the stable G-equivariant homotopy category to the stable motivic homotopy category over k. This functor induces a homomorphism between the endomorphism rings of the unit objects in each category, which are in fact $A(G)$ and $GW(k)$, respectively. In [2, Proposition 3.1], Heller

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and Ormsby show that this homomorphism agrees with $h_{L/k}$. In particular, fullness and faithfulness of $c^*_{L/k}$ are obstructed by surjectivity and injectivity of $h_{L/k}$ respectively.

The main goal of this note is to investigate when the Dress map, and thereby $c^*_{L/k}$, is injective or surjective. While Heller and Ormsby have resolved the investigation when $h_{L/k}$ is an isomorphism [2, Theorem 3.4], we proceed by examining injectivity and surjectivity separately.

When $L = k$ it is obvious that $h_{L/k}$ is injective. The following theorem gives a complete account of when $h_{L/k}$ is injective in the remaining cases.

Theorem 1. *For a finite nontrivial Galois extension L/k, $h_{L/k}$ is injective if and only if $L = k(\sqrt{\alpha})$ where $\alpha \in k^\times$ is not a sum of squares in $k^\times$.*

The proof of Theorem 1 is given in §2. Note that Theorem 1, taken with [2, Proposition 3.1], immediately gives the following corollary.

Corollary 2. *If $c^*_{L/k}$ is faithful, then either L/k is the trivial extension or of the form described in Theorem 1.*

The following theorem gives a complete account of when the Dress map is surjective.

Theorem 3. *For a finite Galois extension L/k, $h_{L/k}$ is surjective if and only if k is quadratically closed in L.*

The proof of Theorem 3 is given in §3. The following corollary is immediate.

Corollary 4. *If $c^*_{L/k}$ is full, then L/k is of the form described in Theorem 3.*

Theorems 1 and 3 combine to replicate Heller and Ormsby's result that for a finite Galois extension L/k, $h_{L/k}$ is an isomorphism if and only if either k is quadratically closed and $L = k$, or k is euclidean and $L = k(i)$. If L/k is the trivial extension then Theorem 3 requires that k be quadratically closed, otherwise Theorem 1 requires that $L = k(\sqrt{\alpha})$ and $k^\times/(k^\times)^2$ contains an element that is not a sum of squares. In the latter case, k must be formally real and then Theorem 3 requires that $k^\times/(k^\times)^2 = \{(k^\times)^2, \alpha(k^\times)^2\}$, *i.e.* $|k^\times/(k^\times)^2| = 2$, so k is euclidean and $\alpha = -1$.

2. Proof of Theorem 1

We begin by stating a number of results that are necessary in the proof of Theorem 1. Many of these results are standard and are stated without proof.

Proposition 5. *Let L/k be a finite Galois extension.*

1. *If $L = k$, then $tr_{L/k}\langle 1\rangle_L = \langle 1\rangle$.*
2. *If $L = k(\sqrt{\alpha})$, then $tr_{L/k}\langle 1\rangle_L = \langle 2, 2\alpha\rangle$.*
3. *If $L = k(\sqrt{\alpha_1}, \sqrt{\alpha_2})$, then $tr_{L/k}\langle 1\rangle_L = \langle 1, \alpha_1, \alpha_2, \alpha_1\alpha_2\rangle$.*

The following is a standard result from Galois theory.

Proposition 6. *Let L/k be a finite Galois extension with Galois group G. If $G \cong \mathbb{Z}/4\mathbb{Z}$, then there is a field E between L and k such that $E = k(\sqrt{\alpha})$ where $\alpha = a^2 + b^2$ for some $a, b \in k^\times$.*

The following theorem is taken directly from Lam [3, Proposition 6.14].