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On the configurations of points in $\mathbb{P}^2$ with the Waldschmidt constant equal to two

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ABSTRACT

Let $Z = \{p_1, \ldots, p_n\}$ be a configuration of points in the projective plane and let $I = I(Z)$ be its corresponding homogeneous ideal in $\mathbb{K}[\mathbb{P}^2]$. In this note, a geometric classification of all configurations of points in $\mathbb{P}^2$, for which the Waldschmidt constant of their defining ideals is equal to two, is given.

1. Introduction

Let $R = \mathbb{K}[\mathbb{P}^N]$ be the coordinate ring of projective space $\mathbb{P}^N$, and let I be a nontrivial homogeneous ideal of R. The m-th symbolic power of the ideal I is defined to be the ideal

$$I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} (R \cap I^m R_P),$$

where the intersection is taken in the field of fractions of R. The Symbolic power of a homogeneous ideal was introduced by W. Krull in commutative algebra and used by Zariski [9] to formulate his fundamental Lemma on holomorphic maps. Motivations for studying symbolic powers of an ideal, arise from different problems in Algebraic Geometry, Commutative Algebra and Combinatorics (see for example [4,7,8]).

Geometrically, if I is a radical ideal and X is the closed subscheme of $\mathbb{P}^N$ determined by I, then $I^{(m)}$ determines a closed subscheme of $\mathbb{P}^N$, whose support is X and consists of all polynomials in R which vanish to order at least m on X. In particular, whenever $X = \{p_1, \ldots, p_n\}$ is a finite set of points in $\mathbb{P}^N$, then the ideal $I = I(X) = I(p_1) \cap \cdots \cap I(p_n)$, where $I(p)$ is the defining ideal of point p, determines a zero dimensional subscheme of $\mathbb{P}^N$ and the ideal $I^{(m)} = I(p_1)^m \cap \cdots \cap I(p_n)^m$ determines a fat points subscheme of $\mathbb{P}^N$, which is convenient to denote it by $mX = mp_1 + \cdots + mp_n$.

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Even though for a general ideal I of R and for every positive integer m, the generators of ordinary powers of I, i.e., I^m, can be obtained from the generators of I in a straightforward manner, the generators of $I^{(m)}$ can not be obtained easily. To find more information on the structure of $I^{(m)}$, the least degree of the minimal generators of $I^{(m)}$ will play a decisive role. For a homogeneous ideal I, let $\alpha(I)$ be the least degree of a minimal generator of I. In [2, Lemma 2.3.1], it is shown that the limit of the sequence $\{\frac{\alpha(I^{(m)})}{m}\}_{m\in\mathbb{N}}$ exists, is denoted by $\widehat{\alpha}(I)$, and is called the Waldschmidt constant of the ideal I. Whenever I is the defining ideal of a closed subscheme $X \subseteq \mathbb{P}^N$, then the Waldschmidt constant of X is defined to be the Waldschmidt constant of I. Moreover, as m tends to infinity, the quotients

$$\frac{\alpha(I^{(m)})}{\alpha(I^m)} = \frac{\alpha(I^{(m)})}{m\alpha(I)} = \frac{\frac{\alpha(I^{(m)})}{m}}{\alpha(I)},$$

tend to $\widehat{\alpha}(I)/\alpha(I)$, which gives an asymptotic measure for the growth of degree of the minimal generators of $I^{(m)}$ compared to degree of the minimal generators of I^m.

Even though $\widehat{\alpha}(I)$ gives an important asymptotic invariant of the closed subscheme determined by I, but computing $\widehat{\alpha}(I)$ is not an easy task and except in a few cases, it is not known for a general homogeneous ideal. Among the known cases, the Waldschmidt constant of the ideal of a finite set of generic points in $\mathbb{P}^N$ [2, Lemma 2.3.1], the Waldschmidt constant of $s \leq (N+1)/2$ general lines in $\mathbb{P}^N$ [5, Theorem 1.5] can be mentioned. The latter fact has been generalized to s general $(t-1)$-dimensional linear subvarieties of $\mathbb{P}^N$, such that $st \leq N+1$ [1, Theorem 7.2]. An interesting result in this direction, is the computation of the Waldschmidt constant of a squarefree monomial ideal I, which is obtained as the optimal solution of a linear program constructed from the primary decomposition of I [1, Theorem 3.2].

A natural question that may arise is to reverse the question of computing the Waldschmidt constant of a radical homogeneous ideal and look for those configurations of points (or linear subspaces) with a given Waldschmidt constant.

In [3], the authors have found and characterized all configurations of points Z in $\mathbb{P}^2$ for which $\widehat{\alpha}(I(Z)) < 9/4$. Motivated by this characterization, we improve this result as follow:

Theorem A. *Let $I(Z)$ be the radical ideal of a finite set of points Z in $\mathbb{P}^2$. Then $\widehat{\alpha}(I(Z)) = 2$ if and only if Z*

a) *consists of $n \geq 4$ points, contained in a smooth conic, or;*
b) *consists of $r+s$ points, where $r, s \geq 2$, and r points out of it lie on a line L_1 and the remaining points lie on another line L_2, or;*
c) *consists of $r+s+1$ points, where $r, s \geq 2$, and r points out of it lie on a line L_1, and s points out of it lie on another line L_2 and a point on the intersection of these two lines, or;*
d) *consists of 6 points formed by the pairwise intersection of 4 lines.*

A step toward proving the **Theorem A**, led us to a result which holds in a more general setting, i.e., for every positive integer d, we construct a large family of configurations of points with Waldschmidt constant equal to d. Specifically we prove:

Theorem B. *For every positive integer d, there exists a configuration of points Z in $\mathbb{P}^2$, such that for every positive integer m, $\alpha(I(Z)^{(m)}) = dm$. In particular, $\widehat{\alpha}(I(Z)) = d$.*

We assume that the ground field $\mathbb{K}$ is algebraically closed of characteristic zero.