Journal of Pure and Applied Algebra 220 (2016) 3826–3860

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Constructing examples of semigroups of valuations

Olga Kashcheyeva

University of Illinois at Chicago, Department of Mathematics, Statistics and Computer Science, 851 S. Morgan (m/c 249), Chicago, IL 60607, USA

A R T I C L E I N F O

Article history:
Received 5 November 2015
Received in revised form 4 April 2016
Available online 8 June 2016
Communicated by A.V. Geramita

A B S T R A C T

We work with rational rank 1 valuations centered in regular local rings of dimension 3. Given a regular local ring R that has an algebraically closed coefficient field k and a zero-dimensional valuation ν, we provide an algorithm for constructing generating sequences for ν in R. We then develop a method for determining a valuation on $k(x, y, z)$ through the sequence of defining values. Using the above results we construct examples of valuations centered in $k[x, y, z]_{(x,y,z)}$ and investigate their semigroups of values.

1. Introduction

This paper is inspired by the following question: given a regular local noetherian domain R and a valuation ν of the quotient field of R dominating R, what semigroups can appear as a value semigroup $\nu(R)$. The answer is available when R is of dimension 1 or 2, but little is known for higher dimensional regular local rings.

The only semigroups which are realized by a valuation on a one dimensional regular local ring are isomorphic to the semigroup of natural numbers. The semigroups which are realized by a valuation on a regular local ring of dimension 2 with algebraically closed residue field are completely classified by Spivakovsky in [15]. A different proof for power series ring in two variables over $\mathbb{C}$ is given by Favre and Jonsson in [8]. In [6], Cutkosky and Vinh give a necessary and sufficient condition for a semigroup S to be the semigroup of a valuation dominating a regular local ring R of dimension 2 with a prescribed residue field extension. In the context of semigroups under the assumption that the rational rank of ν is 1 the criterion is as follows, see [4,2], Corollary 3.3, and [6].

Let S be a well ordered subsemigroup of $\mathbb{Q}_{\geq 0}$ with at most countable system of generators $\{\beta_i\}_{i\geq 0}$ such that $\beta_0 < \beta_1 < \cdots < \beta_n < \ldots$. For all $i \geq 0$ let $G_i = \sum_{j=0}^{i} \beta_j\mathbb{Z}$ and $q_{i+1} = [G_{i+1} : G_i] = \min\{q \in \mathbb{Z}_{>0} | q\beta_{i+1} \in G_i\}$. Then S is the semigroup of a valuation ν dominating a regular local ring R of dimension 2 if and only if $\beta_{i+1} > q_i\beta_i$ for all $i \geq 1$.

E-mail address: kolga@uic.edu.

http://dx.doi.org/10.1016/j.jpaa.2016.05.016
0022-4049/

In particular, it follows that an ordered minimal set of generators $\{\beta_i\}_{i\geq 0}$ of the value semigroup of a valuation dominating a regular local ring of dimension 2 is sparse as $\beta_{i+1} > 2\beta_i$ for all $i \geq 1$. This property does not stand true for higher dimensional regular local rings as shown by example in [2].

When dimension of R is n the classical results, see [19], state that the value semigroup $\nu(R)$ is isomorphic to a well ordered set contained in the nonnegative part of $(\mathcal{R}^h, <_{lex})$ and having an ordinal type of at most ω^h. Here, ω is the first infinite ordinal and h is the rank of ν; h is less than or equal to the rational rank of ν, which is less than or equal to n. Additional bound on the growth of rank 1 valuation semigroups is found by Cutkosky in [1]. It leads to a construction of a well ordered subsemigroup of $\mathbb{Q}_{>0}$ of ordinal type ω, which is not a value semigroup of a noetherian local domain. In [5], Cutkosky and Teissier formulate bounds on the growth of the number of distinct valuation ideals of R corresponding to values lying in certain parts of the value group of ν, thus extending to all ranks the bound given for rank 1 valuations in [1]. They also provide some surprising examples of semigroups of rank greater than 1 that occur as semigroups of valuations on noetherian domains, see [5] and [4]. In [14], Moghaddam constructs a certain class of value semigroups with large rational rank.

Generating sequences of valuations proved to be a very useful tool in understanding valuations centered in two dimensional rings, see [15,3,9,2] and [7] for applications. In this paper we use the approach of generating sequences of valuations to investigate value semigroups of valuations centered in 3-dimensional regular local rings. Let (R, m_R) be a local ring and ν be a valuation of the quotient field of R that dominates R: if (V, m_V) denotes the valuation ring of ν then $R \subset V$ and $m_R = m_V \cap R$. Let $\Phi_R = \nu(R\backslash\{0\})$ be the semigroup consisting of the values of nonzero elements of R. For $\gamma \in \Phi_R$, let $I_\gamma = \{f \in R \mid \nu(f) \geq \gamma\}$ and $I^+_\gamma = \{f \in R \mid \nu(f) > \gamma\}$. A (possibly infinite) sequence $\{Q_i\}$ of elements of R is a *generating sequence* of ν if for every $\gamma \in \Phi_R$ the ideal I_γ is generated by the set

$$\{\prod_i {Q_i}^{b_i} \mid b_i \in \mathbb{Z}_{\geq 0},\ \sum_i b_i\nu(Q_i) \geq \gamma\}.$$

Notice that the set of values $\{\nu(Q_i)\}$ generates Φ_R as a semigroup. Moreover, the set of images of Q_i in the associated graded ring of valuation $\mathrm{gr}_\nu R = \bigoplus_{\gamma\in\Phi} I_\gamma/I^+_\gamma$ generate $\mathrm{gr}_\nu R$ as R/m_R-algebra. The graded ring $\mathrm{gr}_\nu R$ is of particular interest as it is a key tool used by Teissier in [16] and [17] to solve the local uniformization problem. When the valuation is rational, that is $V/m_V = R/m_R$, the graded ring $\mathrm{gr}_\nu R$ is isomorphic to the semigroup algebra over R/m_R of the value semigroup Φ_R, it can be represented as quotient of a polynomial algebra by a prime binomial ideal, (see [17]).

In section 2 we provide an algorithm for constructing generating sequences of zero-dimensional rational rank 1 valuations when R is a 3-dimensional regular local ring that has an algebraically closed coefficient field k. In the construction we denote the sequence $\{P_i\}_{i\geq 0} \cup \{T_i\}_{i>0}$ and call it the sequence of jumping polynomials. We then show that $\{P_i\}_{i\geq 0}\cup\{T_i\}_{i>0}$ is a generating sequence of valuation in section 4. This construction extends the construction of generating sequences in two dimensional regular local rings used in [9].

The algorithm is recursive and explicit equations for P_{i+1} in terms of $\{P_j\}_{0\leq j\leq i}$ and for $T_{\bar{d}(i)}$ in terms of $\{P_j\}_{0\leq j\leq m_i} \cup \{T_j\}_{0<j\leq i}$ are provided. These equations are binomial in nature with the value of the term on the left strictly greater than the value of each term on the right

$$P_{i+1} = P_i^{q_i} - \lambda_i \prod_{j=0}^{i-1} P_j^{n_{i,j}}$$

$$T_{\bar{d}} = T_i^{c_i s_i} \prod_{j=0}^{m_i} P_j^{a_j} \prod_{j=0}^{i-1} T_j^{c_j} - \mu_{\bar{d}} \prod_{j=0}^{m_i} P_j^{n_{\bar{d},j}} \prod_{j=0}^{i-1} T_j^{l_{\bar{d},j}}$$

Here, $\bar{d}$ is an integer greater than or equal to $i+1$, $\lambda_i, \mu_{\bar{d}} \in k \setminus \{0\}$, $n_{i,j}$, m_i, a_j, c_j, $n_{\bar{d},j}$, $l_{\bar{d},j}$ are nonnegative integers and q_i, c_i, s_i are positive integers determined by the algorithm. In the given set