Journal of Pure and Applied Algebra 220 (2016) 3861–3865

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

# Consecutive cancellations in Tor modules over local rings

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A R T I C L E I N F O

*Article history:*
Received 8 November 2015
Received in revised form 8 May 2016
Available online 31 May 2016
Communicated by A.V. Geramita

*MSC:*
Primary: 13D07; secondary: 13A30; 13D02

A B S T R A C T

Let $M$, $N$ be finite modules over a Noetherian local ring $R$. We show that the bigraded Hilbert series of $\mathrm{gr}(\mathrm{Tor}^R(M,N))$ is obtained from that of $\mathrm{Tor}^{\mathrm{gr}(R)}(\mathrm{gr}(M), \mathrm{gr}(N))$ by negative consecutive cancellations, thus extending a theorem of Rossi and Sharifan.



## 1. Introduction

Given a module over a local or graded ring, a major problem in commutative algebra is to understand the behavior of its homological data. It is often convenient to deal with "approximations" of the module with nicer properties, and try to compare their invariants with those of the original module. A well-known phenomenon is the fact that the Betti numbers increase when passing to initial submodules with respect to term orders or weights (cf. [6, Theorem 8.29]), lex-segment submodules (cf. [1,4,5,7]), or associated graded modules with respect to filtrations (cf. [3]). The idea of consecutive cancellations, introduced by Peeva [8] in the graded context, arises in the attempt of describing these inequalities more precisely. Peeva showed that the graded Betti numbers of a homogeneous ideal $I$ in a polynomial ring are obtained from those of the associated lex-segment ideal $\mathrm{Lex}(I)$ by a sequence of *zero consecutive cancellations*, i.e. cancellations from two Betti numbers $\beta_{i+1,j}(\mathrm{Lex}(I)), \beta_{i,j}(\mathrm{Lex}(I))$. The theorem can be generalized to modules (cf. [8, Theorem 1.1] and the subsequent remark).

Now let $(R, \mathfrak{m})$ be a local Noetherian ring and $\mathrm{gr}_{\mathfrak{m}}(R)$ its associated graded ring. A common strategy used to study homological invariants over $R$ is to endow modules and complexes with filtrations and consider the associated graded objects, thus taking advantage of the rich literature available for graded rings. In this framework, Rossi and Sharifan [10,11] investigate the relation between the minimal free resolution of an $R$-module $M$ and that of the associated graded module $\mathrm{gr}_{\mathcal{M}}(M)$ over $\mathrm{gr}_{\mathfrak{m}}(R)$, where $\mathcal{M}$ is the $\mathfrak{m}$-adic filtration or more generally an $\mathfrak{m}$-stable filtration of $M$ (see the next section for definitions). Inspired by

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http://dx.doi.org/10.1016/j.jpaa.2016.05.017
0022-4049/

Peeva's result, they introduce the notion of *negative consecutive cancellations*, i.e. cancellations from two Betti numbers $\beta_{i+1,a}(\mathrm{gr}_{\mathcal{M}}(M)), \beta_{i,b}(\mathrm{gr}_{\mathcal{M}}(M))$ with $a < b$. They prove the following result for the case of a regular local ring:

**Theorem 1** *([11, Theorem 3.1]). Let $(R, \mathfrak{m})$ be a regular local ring and $\mathrm{gr}_{\mathfrak{m}}(R)$ its associated graded ring. If $M$ is a finite $R$-module with an $\mathfrak{m}$-stable filtration $\mathcal{M}$, the Betti numbers of $M$ are obtained from the graded Betti numbers of the $\mathrm{gr}_{\mathfrak{m}}(R)$-module $\mathrm{gr}_{\mathcal{M}}(M)$ by a sequence of negative consecutive cancellations.*

Their argument relies on explicit manipulations on the matrices of the differential in a free $R$-resolution of $M$ obtained by lifting the minimal free $\mathrm{gr}_{\mathfrak{m}}(R)$-resolution of $\mathrm{gr}_{\mathcal{M}}(M)$. In this note we take an alternative approach, namely we employ a spectral sequence, introduced by Serre [12], arising from a suitable filtered complex. We provide the following generalization of Theorem 1, which holds for Tor modules over an arbitrary Noetherian local ring:

**Theorem 2.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\Bbbk = R/m$ its residue field, and $\mathrm{gr}_{\mathfrak{m}}(R)$ its associated graded ring. Given two finite $R$-modules $M$, $N$ with $\mathfrak{m}$-stable filtrations $\mathcal{M}$, $\mathcal{N}$, there exist $\mathfrak{m}$-stable filtrations on each $\mathrm{Tor}_i^R(M, N)$ such that the Hilbert series*

$$\sum_{i,j} \dim_{\Bbbk} \mathrm{gr}(\mathrm{Tor}_i^R(M, N))_j z^i t^j$$

*is obtained from the Hilbert series*

$$\sum_{i,j} \dim_{\Bbbk} \mathrm{Tor}_i^{\mathrm{gr}_{\mathfrak{m}}(R)}(\mathrm{gr}_{\mathcal{M}}(M), \mathrm{gr}_{\mathcal{N}}(N))_j z^i t^j$$

*by negative consecutive cancellations, i.e. subtracting terms of the form $z^{i+1}t^a + z^i t^b$ with $a < b$.*

## 2. Proof of the main result

We begin by providing the necessary background and definitions. Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field $\Bbbk = R/\mathfrak{m}$. The ring $G = \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{n \geq 0} \mathfrak{m}^j/\mathfrak{m}^{j+1}$ is known as the *associated graded ring* of $R$, and it is a standard graded $\Bbbk$-algebra. Geometrically, if $R$ is the local ring of a variety $\mathcal{V}$ at a point $\mathfrak{p}$, then $G$ is the homogeneous coordinate ring of the tangent cone to $\mathcal{V}$ at $\mathfrak{p}$; for this reason $G$ is sometimes called the tangent cone of $R$.

Let $M$ be a finitely generated $R$-module. A descending filtration $\mathcal{M} = \{M^j\}_{j \in \mathbb{N}}$ of $R$-submodules of $M$ is said to be $\mathfrak{m}$-*stable* if the following conditions are satisfied: $M^0 = M$, $\mathfrak{m}M^j \subseteq M^{j+1}$ for all $j \geq 0$, and $\mathfrak{m}M^j = M^{j+1}$ for sufficiently large $j$. If $\mathcal{M}$ is an $\mathfrak{m}$-stable filtration of $M$, we define the *associated graded module of $M$ with respect to $\mathcal{M}$* as $\mathrm{gr}_{\mathcal{M}}(M) = \bigoplus_{j \geq 0} M^j/M^{j+1}$; it is a finitely generated graded $G$-module. The central example is that of the $\mathfrak{m}$-adic filtration $\mathcal{M} = \{\mathfrak{m}^j M\}_{j \in \mathbb{N}}$, however it will be necessary to allow for more general ones. In order to simplify the notation, sometimes we will just write $\mathrm{gr}(M)$ if the filtration is clear from the context.

Let $M$, $N$ be two $R$-modules with $\mathfrak{m}$-stable filtrations. An $R$-linear map $f : M \to N$ such that $f(M^j) \subseteq N^j$ for all $j$ induces a homogeneous $G$-linear map $\mathrm{gr}(f) : \mathrm{gr}(M) \to \mathrm{gr}(N)$. In fact, $\mathrm{gr}(\cdot)$ is a functor from the category of $R$-modules with $\mathfrak{m}$-stable filtrations to the category of graded $G$-modules, and in particular we can consider associated graded complexes of filtered complexes. An important result due to Robbiano [9] states that it is possible to "lift" a graded free $G$-resolution $\mathbf{G}$ of $\mathrm{gr}_{\mathcal{M}}(M)$ to a free $R$-resolution $\mathbf{F} = \{F_i\}$ of $M$ together with $\mathfrak{m}$-stable filtrations $\{F_i^j\}_{j \in \mathbb{N}}$ on each free module $F_i$ such that $\mathrm{gr}(\mathbf{F}) = \mathbf{G}$. The lifted