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Schemes over symmetric monoidal categories and torsion theories

Abhishek Banerjee

Institut des Hautes Études Scientifiques, Le Bois-Marie 35, route de Chartres, 91440, Bures-sur-Yvette, France

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A B S T R A C T

Let $(\mathcal{C}, \otimes, 1)$ be an abelian symmetric monoidal category satisfying certain conditions and let X be a scheme over $(\mathcal{C}, \otimes, 1)$ in the sense of Toën and Vaquié. In this paper, we construct torsion theories on the categories $\mathcal{O}_X$-*Mod* and $QCoh(X)$ respectively of $\mathcal{O}_X$-modules and quasi-coherent sheaves on X, when X is Noetherian and integral over $(\mathcal{C}, \otimes, 1)$. Thereafter, we study these torsion theories with respect to the quasi-coherator $Q_X : \mathcal{O}_X\text{-}Mod \longrightarrow QCoh(X)$ that is right adjoint to the inclusion $i_X : QCoh(X) \longrightarrow \mathcal{O}_X\text{-}Mod$. Finally, we obtain an alternative description of the quasi-coherator $Q_X(\mathcal{F})$ as a subsheaf of $\mathcal{F}$, when $\mathcal{F} \in \mathcal{O}_X$-*Mod* satisfies certain conditions. Along the way, we present further results on the notions of "Noetherian" and "integral" for schemes over $(\mathcal{C}, \otimes, 1)$ that we believe to be of independent interest.

1. Introduction

Let $(\mathcal{C}, \otimes, 1)$ be an abelian symmetric monoidal category satisfying certain conditions. The idea of doing algebraic geometry over a symmetric monoidal category has been pursued by several authors (see, for instance, Deligne [7], Hakim [12], Toën and Vaquié [22]). When $\mathcal{C}$ is chosen to be k-*Mod*, the category of modules over a commutative ring k, it reduces to the ordinary algebraic geometry of schemes over $Spec(k)$.

In this paper, we continue from [1,3,4] our effort to carry out a systematic study of quasi-coherent sheaves for schemes over $(\mathcal{C}, \otimes, 1)$. More specifically, we investigate torsion theories for the categories $\mathcal{O}_X$-*Mod* and $QCoh(X)$ of $\mathcal{O}_X$-modules and quasi-coherent sheaves respectively for a scheme X over $(\mathcal{C}, \otimes, 1)$ in the sense of Toën and Vaquié [22]. Torsion theories on an abelian category are closely related to t-structures (see [6,13]): a torsion theory on an abelian category induces a t-structure on its bounded derived category. In turn, the heart of this t-structure gives us a new abelian category, also with its own t-structure and so on. Such iterations are a key tool for studying abelian categories; for instance, they have been used to recover Deligne's work defining perverse t-structures by means of tilting with respect to torsion theories (see [24]). The study of torsion theories is also closely linked to recollements of abelian categories (see [6]) and we hope

E-mail address: abhishekbanerjee1313@gmail.com.

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that this present work will help us to study recollements of the category $QCoh(X)$ for a scheme X over $(\mathcal{C},\otimes,1)$. As such, in this paper, we will construct hereditary torsion theories of finite type on $\mathcal{O}_X$-*Mod* and $QCoh(X)$ when X is a Noetherian integral scheme over $(\mathcal{C},\otimes,1)$. Further, we will study how these torsion theories behave with respect to the quasi-coherator $Q_X : \mathcal{O}_X\text{-}Mod \longrightarrow QCoh(X)$ constructed in [3]. As an application, we shall also obtain by an alternative method an analogue of a recent result of Odabaşi [18] on the product of quasi-coherent sheaves. Along the way, we will develop the notions of "Noetherian schemes" and "integral schemes" over $(\mathcal{C},\otimes,1)$ and demonstrate several of their properties that we believe to be of independent interest.

We will now describe the paper in more detail. In order to work with torsion theories in the context of schemes over a symmetric monoidal category, we will first need to extend some commutative algebra to $(\mathcal{C},\otimes,1)$. This is done in Section 2. We denote by $Comm(\mathcal{C})$ the category of commutative monoid objects in $(\mathcal{C},\otimes,1)$. It is clear that in order to consider torsion submodules of an A-module for some $A \in Comm(\mathcal{C})$, we need to work with "integral monoid objects". As such, we will say that $A \in Comm(\mathcal{C})$ is integral if $\mathcal{E}(A) := Hom_{A\text{-}Mod}(A,A)$ is an ordinary integral domain (see Definition 2.1). The key tool in Section 2 will be the method of localization of commutative monoid objects in $(\mathcal{C},\otimes,1)$ that we have developed in [2]. More precisely, for an integral commutative monoid object $A \in Comm(\mathcal{C})$, we will consider the "field of fractions" $K(A)$:

$$K(A) := \underset{f\in\mathcal{E}(A)\setminus\{0\}}{colim} A_f \tag{1.1}$$

where A_f is the localization of A with respect to $f \in \mathcal{E}(A)$ as defined in [2, §3]. Then, the main results of Section 2 may be summarized as follows.

Theorem 1.1. *Let A be an integral commutative monoid object in $(\mathcal{C},\otimes,1)$. Then:*

(a) If M is an A-module, the torsion submodule $T_A(M)$ of M may be expressed as the kernel of the canonical morphism:

$$T_A(M) \cong Ker(M \xrightarrow{\cong} M\otimes_A A \longrightarrow M\otimes_A K(A)) \tag{1.2}$$

Further, M is a torsion A-module if and only if $M\otimes_A K(A) = 0$.

(b) Let $\mathfrak{T}_A$ (resp. $\mathfrak{F}_A$) denote the full subcategory of torsion modules (resp. torsion free modules) in A-Mod. Then, the pair $(\mathfrak{T}_A,\mathfrak{F}_A)$ forms a hereditary torsion theory of finite type on A-Mod.

In Section 3, we start working with schemes over $(\mathcal{C},\otimes,1)$. Let $Aff_{\mathcal{C}} := Comm(\mathcal{C})^{op}$ be the category of affine schemes over $(\mathcal{C},\otimes,1)$. For any $A \in Comm(\mathcal{C})$, we denote by $Spec(A)$ the affine scheme corresponding to A. We then recall from Toën and Vaquié [22] the definition of a scheme X over the symmetric monoidal category $(\mathcal{C},\otimes,1)$ (see Definition 3.1). We denote by $ZarAff(X)$ the category of Zariski open immersions $U \longrightarrow X$ with $U = Spec(A)$ affine. We also recall briefly in Definition 3.2 the notion of a sheaf of $\mathcal{O}_X$-modules for a scheme X over $(\mathcal{C},\otimes,1)$ that was introduced in [3]. Now, as mentioned before, it is clear from (1.2) that $K(A)$ plays the role of the "field of fractions" for an integral monoid object A in $(\mathcal{C},\otimes,1)$. However, our notion of "integral", where $A \in Comm(\mathcal{C})$ is said to be an integral monoid object if $Hom_{A\text{-}Mod}(A,A)$ is an integral domain, is really "at the level of global sections". Therefore, in order to make sure that $K(A)$ satisfies many more field like properties, we will need to make two further assumptions. The first of these is that from Section 3 onwards, we will only work with commutative monoid objects that are also "Noetherian" (see Definition 3.5), i.e., any finitely generated A-module M may be expressed as a colimit

$$M \cong colim(0 \longleftarrow A^m \xrightarrow{q} A^n) \tag{1.3}$$