Journal of Pure and Applied Algebra 220 (2016) 3067–3087

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Coherence conditions in flat regular pullbacks

Jason Boynton *, Sean Sather-Wagstaff [1,2]

Department of Mathematics, North Dakota State University Department # 2750, PO Box 6050, Fargo, ND 58108-6050, USA

ARTICLE INFO

Article history:
Received 3 November 2015
Received in revised form 2 February 2016
Available online 15 March 2016
Communicated by A.V. Geramita

MSC:
13A99; 13B40; 13C11; 13D07; 13F20; 13G05

ABSTRACT

We investigate the behavior of four coherent-like conditions in regular conductor squares. In particular, we find necessary and sufficient conditions in order that a pullback ring be a finite conductor ring, a coherent ring, a generalized GCD ring, or quasi-coherent ring. As an application of these results, we are able to determine exactly when the ring of integer-valued polynomials determined by a finite subset possesses one of the four coherent-like properties.

© 2016 Elsevier B.V. All rights reserved.

1. Introduction

Throughout this paper, the term "ring" is short for "commutative ring with identity", and "module" is short for "unital module".

The last half century has seen an abundance of research dedicated to the study of the transference of various ring and ideal-theoretic properties in pullback constructions. It is well known that these pullback constructions provide a rich source of (counter)examples in commutative algebra; see the foundational paper [8] and the survey [17]. In his book [12], Gilmer popularized a very special case of a pullback called the $D + M$ construction. Gilmer's construction begins with a valuation domain V containing a retract field K, meaning that $V = K + M$ for some maximal ideal M of V. Let D be a subring of K, and form the subring $D + M \subset V$. In [7], Dobbs and Papick find necessary and sufficient conditions on K and D (or M) in order that $D + M$ is a coherent ring. In [3], Brewer and Rutter dropped the valuation condition on T and found similar conditions on the constituent rings so that the ring $D + M$ is coherent. In [11], Houston

* Corresponding author.
E-mail addresses: jason.boynton@ndsu.edu (J. Boynton), ssather@clemson.edu (S. Sather-Wagstaff).
URLs: http://math.ndsu.nodak.edu/faculty/boynton.shtml (J. Boynton), http://people.clemson.edu/~ssather/ (S. Sather-Wagstaff).
[1] Current address: Department of Mathematical Sciences, Clemson University, O-110 Martin Hall, Box 340975, Clemson, SC 29634, USA.
[2] Sean Sather-Wagstaff was supported in part by NSA grant H98230-13-1-0215.

http://dx.doi.org/10.1016/j.jpaa.2016.02.004
0022-4049/© 2016 Elsevier B.V. All rights reserved.

and Gabelli offer improved results on the transference of coherence and other coherent-like conditions by removing the assumption that the domain T contains a retract field.

The purpose of this article is to investigate the transference of coherent-like conditions in a more general setting that is very similar to the pullback construction of [10]; see Problem 1.1. We are primarily concerned with finding conditions under which the (quasi-)coherent, finite conductor, and generalized GCD properties ascend and descend in a pullback. The construction central to our study, is described here explicitly. Start with a ring surjection $\eta_1 \colon T \twoheadrightarrow B$ and an inclusion of rings $\iota_1 \colon A \hookrightarrow B$ with $B \neq 0$, hence $A \neq 0$. Let R denote the pullback of these maps, that is, the subring of $A \times T$ consisting of all elements (a, t) such that $\iota_1(a) = \eta_1(t)$. The natural maps $\eta_2 \colon R \twoheadrightarrow A$ and $\iota_2 \colon R \hookrightarrow T$ yield a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} R & \overset{\iota_2}{\hookrightarrow} & T \\ {\scriptstyle \eta_2}\downarrow & & \downarrow{\scriptstyle \eta_1} \\ A & \underset{\iota_1}{\hookrightarrow} & B \end{array} \qquad (\square)$$

such that $\mathrm{Ker}(\eta_2) = \mathrm{Ker}(\eta_1)$. The common ideal $\mathrm{Ker}(\eta_i)$ is the largest common ideal of R and T; it is denoted C and called the *conductor* of T into R. When C contains a T-regular element, we say that the conductor square $(\square)$ is *regular*.

Conductor squares can also be built as follows [4]. Let T be a commutative ring with subring R, and suppose that R and T have a common, non-zero ideal. We call the largest common ideal C the *conductor* of T into R. Setting $A = R/C$ and $B = T/C$, we obtain a commutative diagram $(\square)$ which is a conductor square.

It is common in the study of pullback constructions to assume that T is an integral domain and that C is a maximal ideal of T. However, important examples are obtained by allowing zero-divisors in the pullback square. For example, let D be an integral domain with field of fractions K, and let $E = \{e_1, \ldots, e_r\} \subset D$. Setting $T = K[X]$ and $C = (X - e_1) \cdots (X - e_r)K[X]$, we have $B = T/C \cong \prod_{i=1}^{r} K$. Using $A = \prod_{i=1}^{r} D$ in the conductor square, we get

$$R = \mathrm{Int}(E, D) = \{g \in K[X] \mid g(E) \subset D\}$$

the ring of integer-valued polynomials on D determined by the subset E; see [5]. Observe that the rings A and B are not integral domains. In fact, Chapman and Glaz have proposed the following open question.

Problem 1.1. (See [6, Problem 50].) Study the ring and ideal-theoretic properties that transfer in a conductor square where the conductor ideal is not maximal (or even prime) in the extension ring.

This paper is organized as follows. In Section 2, we provide the relevant definitions and some background results for pullbacks in which the extension $\iota_2 \colon R \hookrightarrow T$ is flat. In Section 3, we show that, in a non-trivial pullback square, the conductor C is *never* finitely generated over R, and $\iota_1 \colon A \hookrightarrow B$ is *never* a faithfully flat extension of rings, whenever $\iota_2 \colon R \hookrightarrow T$ is flat. In light of [2,11], this suggests that if R is a coherent-like ring defined by a regular conductor square of the type $(\square)$, then either T is R-flat or C (and T) is finitely generated over R.

The remaining sections, we assume that T is R-flat in the regular conductor square $(\square)$. In Section 4, we assume that the conductor C is principal in T and that a unitary type of condition similar to [19] holds in order to find necessary and sufficient conditions on the constituent rings A and T so that R is a finite conductor ring. The proofs of these results all extend naturally to the coherent ring case. We conclude the section by showing that if E is finite, then the ring $\mathrm{Int}(E, D)$ is a finite conductor ring if and only if D is a finite conductor ring. We note that a similar result holds for $\mathrm{Int}(E, D)$ in the coherent case. In Sections 5

Download English Version:

https://daneshyari.com/en/article/4595768

Download Persian Version:

https://daneshyari.com/article/4595768

Daneshyari.com