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Canonical modules of complexes

Maryam Akhavin, Eero Hyry *

Mathematics and Statistics, School of Information Sciences, University of Tampere, FIN-33014 Tampereen yliopisto, Finland

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ABSTRACT

We define the notion of the canonical module of a complex. We then consider Serre's conditions for a complex and study their relationship to the local cohomology of the canonical module and its ring of endomorphisms.

1. Introduction

The notion of the canonical module of a ring is an important tool in commutative algebra. P. Schenzel defined in [1] the notion of the canonical module of a module. The canonical module always satisfies Serre's condition (S_2). Schenzel related higher Serre's conditions to the vanishing of certain local cohomology modules of the canonical module. The purpose of this article is to extend these results to complexes. We utilize the powerful tools of hyperhomological algebra.

Let (R, m) be a Noetherian local ring admitting a dualizing complex. We work within the derived category $D_b^f(R)$ of bounded complexes of R-modules with finitely generated homology. Generalizing the work of Schenzel, we define for any complex $M \in D_b^f(R)$ and any $i \in \mathbb{Z}$ the i-th module of deficiency K_M^i by setting $K_M^i = \mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(M, D_R))$, where D_R denotes the dualizing complex of R normalized so that $\mathrm{H}_{\dim R}(D_R) \neq 0$ and $\mathrm{H}_i(D_R) = 0$ for $i > \dim R$. The canonical module of M is then $K_M = K_M^{\dim M}$. Note that by local duality the local cohomology module $\mathrm{H}_m^i(M)$ is the Matlis dual of K_M^i. In particular, modules of deficiency measure how far the complex is from being Cohen–Macaulay.

Given $k \in \mathbb{N}$, we say that a complex M satisfies Serre's condition (S_k) if

$$\operatorname{depth}_{R_p} M_p \geq \min(k - \inf M_p, \dim_{R_p} M_p)$$

* Corresponding author.
E-mail addresses: maryam.akhavin@uta.fi (M. Akhavin), eero.hyry@uta.fi (E. Hyry).

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for all prime ideals $p \in \operatorname{Supp}_R M$. It is convenient to consider complexes satisfying the condition $\dim_R M = \dim_{R_p} M_p + \dim R/p$ for every $p \in \operatorname{Supp}_R M$. Here $\operatorname{Supp}_R M$ means the homological support of M. It then follows from our Theorem 4.12 that (S_k) is equivalent to the natural homomorphism

$$\operatorname{Ext}_R^{-i}(M, M) \to K^{i+\dim M}_{M \otimes^L_R K_M}$$

being bijective for all $i \geq -k+2$, and injective for $i = -k+1$. Note that $K_{M \otimes^L_R K_M} = \operatorname{Hom}_R(K_M, K_M)$. It makes also sense, for any $l \in \mathbb{Z}$, to look at the condition $(S_{k,l})$ saying that

$$\operatorname{depth}_{R_p} M_p \geq \min(k - l, \dim M_p)$$

for all prime ideals $p \in \operatorname{Supp}_R M$. Observe that (S_k) always implies $(S_{k,\sup M})$. It now turns out in Corollary 4.14 that $(S_{k,l})$ is equivalent to the natural homomorphism $\mathrm{H}_i(M) \to K^{i+t}_{K_M}$ being bijective for $i \geq l-k+2$, and injective for $i = l-k+1$. In the case M is a module and $l = 0$, this reduces to the result of Schenzel mentioned in the beginning.

Finally, we look at the complex $M^\dagger := \mathbf{R}\operatorname{Hom}_R(M, D_R)$. Suppose that $\sup M = \sup M_p$ for all $p \in \operatorname{Supp} M$. If $M^\dagger$ satisfies Serre's condition (S_2), it comes out in Corollary 4.22 that $K_M \cong K_{\mathrm{H}_s(M)}$, where $s = \sup M$. Combining this with the observation of Lipman, Nayak and Sastry in [2, Proposition 9.3.5] that the Cousin complex of a complex depends only on the top homology of $M^\dagger$ i.e. on the canonical module, we can relate the Cousin complex of the complex M to that of the module $\mathrm{H}_s(M)$. More precisely, we show in Proposition 4.24 that in the above situation

$$E_{\mathcal{D}(M)}(M) \cong \sum\nolimits^s E_{\mathcal{D}(\mathrm{H}_s(M))}(\mathrm{H}_s(M)),$$

where $\mathcal{D}(M)$ and $\mathcal{D}(\mathrm{H}_s(M))$ denote the dimension filtrations of M and $\mathrm{H}_s(M)$ respectively.

2. Preliminaries

The purpose of this section is to fix notation and recall some definitions and results of hyperhomological algebra relevant to this article. As a general reference, we mention [3] and references therein. For more details, see also [4–7].

In the following R is always a commutative Noetherian ring. If R is local, then m denotes the maximal ideal and k the residue field of R.

Throughout this article we work within the derived category $D(R)$ of R-modules. We use homological grading so that the objects of $D(R)$ are complexes of R-modules of the form

$$M: \quad \dots \xrightarrow{d_{i-1}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \dots \ .$$

The derived category is triangulated, the suspension functor Σ being defined by the formulas $(\Sigma M)_n = M_{n-1}$ and $d_n^{\Sigma M} = -d_{n-1}$. The symbol "$\simeq$" is reserved for isomorphisms in $D(R)$. We use the subscripts "b", "$+$" and "$-$" to denote the homological boundness, the homological boundness from below and the homological boundness from above, respectively. The superscript "f" denotes the homological finiteness. So the full subcategory of $D(R)$ consisting of complexes with finitely generated homology modules is denoted by $D^f(R)$. As usual, we identify the category of R-modules as the full subcategory of $D(R)$ of complexes M satisfying $\mathrm{H}_i(M) = 0$ for $i \neq 0$. For a complex $M \in D(R)$, by $\sup M$ and $\inf M$, we mean its homological supremum and infimum. Let M and N be complexes of R-modules. We use the standard notations $M \otimes^L_R N$ and $\mathbf{R}\operatorname{Hom}_R(M, N)$ for the left derived tensor product complex and the right derived homomorphism complex, respectively. Moreover, we set $\operatorname{Ext}_R^{-i}(M, N) = \mathrm{H}_i(\mathbf{R}\operatorname{Hom}_R(M, N))$ for all $i \in \mathbb{Z}$.