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Invariants of a free linear category and representation type [☆]

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We consider a homogeneous action of a finite group on a free linear category over a field in order to prove that the subcategory of invariants is still free. Moreover we show that the representation type is preserved when considering invariants.

1. Introduction

The first purpose of this article is to prove that the category of invariants of a homogeneous action of a finite group on a free linear category is again a free linear category. V.K. Kharchenko [15] and D.R. Lane [16] proved that the algebra of invariants of a finite group acting homogeneously on a free algebra is a free algebra.

In this paper k is a field of any characteristic. A k-category is a small category enriched over k-vector spaces, see for instance [14]. In other words objects are a set, morphisms are vector spaces and composition is k-bilinear. At each object the endomorphisms form a k-algebra.

A free k-category is given by a set of objects and a set of "oriented" vector spaces that is, to each couple of objects there is a given vector space which can be zero. The *track-quiver* of this data is the oriented graph which records the non-zero vector spaces. In the first section we recall the precise definitions and the construction of the associated free k-category. We also recall that if the number of objects of a k-category is finite, there is a canonical k-algebra associated to it. This algebra is hereditary if the category is free.

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The problem of describing invariants of an action of a group goes back to D. Hilbert and E. Noether. For commutative free algebras (*i.e.* polynomial algebras) over a field of characteristic zero, a homogeneous action of a finite subgroup of $GL(d,k)$ provides again a commutative free algebra of invariants if and only if the group is generated by pseudo-reflections. This has been proved by G.C. Shephard and J.A. Todd [21], and by C. Chevalley and J.P. Serre [4,20]. For a detailed account see [22,2].

The action of a group on a free k-category is called homogeneous if it is given by a linear action on the generators, which is extended in the unique possible manner to an action on the free k-category by invertible endofunctors. Note that we consider actions on a free k-category which are trivial on the set of objects. Indeed, we restrict to this case since otherwise the invariants are the invariants of the full subcategory determined by the invariant objects. Observe that this context is at the opposite of a Galois action of a group on a k-category where the action on objects has to be free, hence with no invariant objects, see for instance [6].

Another aim of this paper is to prove a result about invariants of a tensor product of kG-modules which appears to be unknown. This result is a crucial tool for proving our main theorem, namely that invariants of a homogeneous action on a free k-category form again a free k-category. It can be outlined as follows. Let $M_n, \dots, M_1$ be a sequence of kG modules and let $(M_n \otimes \cdots \otimes M_1)^G$ be the vector space of invariants of the action of G on the tensor product. Some of these invariants "are strictly from below" in the sense that they are sums of tensors of invariants of a partition by strict sub-strings of $M_n \otimes \cdots \otimes M_1$. We call those invariants "composites", they form a canonical sub-vector space of the space of invariants. An arbitrarily chosen vector space complement is always called a space of irreducible invariants. For any fixed choice of irreducible spaces of invariants in the sub-strings of the tensor product, we prove the uniqueness of the decomposition of an invariant as a sum of tensors of irreducible invariants. We infer this result from the Kharchenko–Lane Theorem.

The last section is also a contribution to the classical study of the relationship of an algebra with its algebra of invariants, see for instance [19]. We prove that the representation type is conserved, more precisely if a free k-linear category with a homogeneous action of a finite group is of finite or tame representation type, then its invariant k-category is respectively of finite or tame representation type. Note that we adopt the convention that a k-category of finite representation type is also of tame representation type. In order to prove this result, we set up ad hoc cleaving techniques whose basics goes back to D.G. Higman.

At least two questions arise from the present work in relation with previous results for free algebras.

Firstly V.O. Ferreira, L.S.I. Murakami and A. Paques in [18] have provided a generalization of the work of V.K. Kharchenko by considering a homogeneous action of a Hopf k-algebra over a free k-algebra. A generalization of our work should also hold for a Hopf algebra acting homogeneously on a free k-category.

Secondly note that W. Dicks and E. Formanek [7] have proved that for a free k-algebra, unless the group is cyclic and the number of free generators of the original algebra is finite, the number of free generators of the invariants is infinite. A similar result in the setting of free k-categories should hold.

2. Free linear categories

We recall first the definition of a free linear category $\mathcal{L}_k(V)$ over a field k which satisfies the universal property stated in Proposition 2.2. For this purpose, let $\mathcal{L}_0$ be a set and let $V = \{{}_yV_x\}_{y,x\in\mathcal{L}_0}$ be a family of k-vector spaces. Let $u = (u_n, \cdots, u_0)$ be a sequence of elements in $\mathcal{L}_0$ and consider the vector space

$$W(u) = {}_{u_n}V_{u_{n-1}} \otimes \cdots \otimes {}_{u_2}V_{v_1} \otimes {}_{u_1}V_{u_0}.$$

For a singleton sequence $u = (u_0)$ we set $W(u_0) = k$.