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Derived invariants for surface algebras

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ABSTRACT

In this paper we study the derived equivalences between *surface algebras*, introduced by David-Roesler and Schiffler [11]. Each surface algebra arises from a cut of an ideal triangulation of an unpunctured marked Riemann surface with boundary. A cut can be regarded as a grading on the Jacobian algebra of the quiver with potential (Q, W) associated with the triangulation.
Fixing a set ϵ of generators of the fundamental group of the surface $\pi_1(S)$, we associate to any cut d a weight $w^\epsilon(d) \in \mathbb{Z}^{2g+b}$, where g is the genus of S and b the number of boundary components. The main result of the paper asserts that the derived equivalence class of the surface algebra is determined by the corresponding weight $w^\epsilon(d)$ up to homeomorphism of the surface. Surface algebras are gentle and of global dimension ≤ 2, and any surface algebras coming from the same surface (S, M) are cluster equivalent, in the sense of [2]. To prove that the weight is a derived invariant we strongly use results about cluster equivalent algebras from [2]. Furthermore we also show that for surface algebras the invariant defined for gentle algebras by Avella-Alaminos and Geiss in [6] is determined by the weight.

1. Introduction

In this paper we study derived equivalence for the class of algebras called surface algebras. Surface algebras were introduced by David-Roesler and Schiffler in [11].

Let S be an oriented Riemann surface with non-empty boundary, and M be a set of marked points on the boundary of S. A quiver with potential (Q^Δ, W^Δ) is associated to each ideal triangulation Δ of (S, M) in [5]. The potential consists of the sum of the oriented three cycles in the quiver which correspond to internal triangles of the triangulation. The Jacobian algebra of (Q^Δ, W^Δ), denoted $\mathrm{Jac}(Q^\Delta, W^\Delta)$, is a finite dimensional gentle algebra [5].

A surface algebra is constructed from an ideal triangulation by introducing cuts in the internal triangles [11]. This yields a grading on $\mathrm{Jac}(Q^\Delta, W^\Delta)$, and the surface algebra $\Lambda = (\Delta, d)$ associated with the

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grading is the subalgebra of degree zero of $\mathrm{Jac}(Q^\Delta, W^\Delta)$. By [11], surface algebras are finite dimensional, gentle and have global dimension ≤ 2.

A cluster category $\mathcal{C}_{(Q,W)}$ was associated to an arbitrary Jacobi-finite quiver with potential in [3]. For a surface (S, M) the category $\mathcal{C}_{(Q^\Delta,W^\Delta)}$ does not depend on the ideal triangulation Δ [18,17]. The cluster category of a marked surface without punctures has been studied in [8], where the Auslander–Reiten structure of $\mathcal{C}_{(S,M)}$ was described using a geometric model.

In [3] a cluster category is also associated to any finite dimensional algebra of global dimension 2 satisfying a certain homological property (τ_2-finiteness). We show in section 3.2 that when $\Lambda = (S, M, \Delta, d)$ is a surface algebra then Λ is τ_2-finite and the cluster category of Λ is equivalent to the cluster category of (Q^Δ, W^Δ). Thus any two surface algebras coming from the same surface are cluster equivalent, in the sense of [2].

Let g be the genus of S, and b the number of boundary components of S. We associate a weight $w^\epsilon(d)$ in $\mathbb{Z}^{2g+b}$ to each cut d and set ϵ of generators of $\pi_1(S)$. Then using results in [2] we are able to prove our main theorem.

Theorem. *Let (S, M) be a surface which is not a disc and let Λ and Λ' be the surface algebras associated with (Δ, d) and (Δ', d'), respectively. Then the following statements are equivalent:*

(1) $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\Lambda')$.
(2) *There exists an orientation preserving homeomorphism $\Phi : S \to S$ with $w^\epsilon(\Delta, d) = w^\epsilon(\Phi^{-1}(\Delta'), d' \circ \Phi)$ (or equivalently $w^\epsilon(\Delta, d) = w^{\Phi(\epsilon)}(\Delta', d')$).*

In the case where the surface is a disc with marked points on the boundary, it has already been shown in [4] that all surface algebras are derived equivalent.

Our result also generalizes a result in [10], in which the author studies the case where $\Delta = \Delta'$ and the genus of S is zero.

As mentioned above surface algebras are gentle, see [11]. Avella-Alaminos and Geiss (abbreviated AG) introduced a derived invariant for gentle algebras in [6]. The invariant determines derived equivalence for gentle algebras with at most one cycle in the quiver. This invariant is calculated combinatorially using the quiver and relations of the algebra. In [11] the AG-invariant of any surface algebra is computed using the ideal triangulation of the surface and the cut. Using this description, we show that the AG-invariant is determined by the weight of the corresponding cut. Our theorem then implies that in the genus zero case, the AG invariant determines the derived equivalence class of the surface algebra. In particular, it is easy to find derived equivalent surface algebras with more than one cycle in the quiver. Thus our result expands the class of algebras where the AG-invariant determines derived equivalence.

The paper is organized as follows. In section 2 we give the topological background and define the weight of a grading. In section 3 we give the relevant background on graded Jacobian algebras and generalized cluster categories before we state and prove our main theorem in subsection 3.4. In section 4 we examine the genus 0 case and give examples. Finally, in section 5 we study the relationship between the weight and the AG-invariant.

2. Surfaces without punctures and quivers

In this section we first recall how to find a quiver from an ideal triangulation in section 2.1. We give the definition of weight in section 2.2, and in section 2.3 we look at how mutation affects the weight.

2.1. Ideal triangulations

We recall here some definitions of [13].