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Eilenberg–Watts Theorem for 2-categories and quasi-monoidal structures for module categories over bialgebroid categories

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ABSTRACT

We prove Eilenberg–Watts Theorem for 2-categories of the representation categories $\mathcal{C}$- Mod of finite tensor categories $\mathcal{C}$. For a consequence we obtain that any autoequivalence of $\mathcal{C}$- Mod is given by tensoring with a representative of some class in the Brauer–Picard group $\mathrm{BrPic}(\mathcal{C})$. We introduce bialgebroid categories over $\mathcal{C}$ and a cohomology over a symmetric bialgebroid category. This cohomology turns out to be a generalization of the one we developed in a previous paper and moreover, an analogous Villamayor–Zelinsky sequence exists in this setting. In this context, for a symmetric bialgebroid category $\mathcal{A}$, we interpret the middle cohomology group appearing in the third level of the latter sequence. We obtain a group of quasi-monoidal structures on the representation category $\mathcal{A}$- Mod.

1. Introduction

In our previous paper [15] we introduced a cohomology over a symmetric finite tensor category $\mathcal{C}$ and we constructed an infinite exact sequence a la Villamayor–Zelinsky which contains three types of cohomology groups. These three types of groups repeat periodically in the sequence and we consider the sequence so that in each level there are three groups of different types. In [16] we interpreted the middle term in the second level of this sequence, obtaining the group of quasi $\mathcal{C}$-coring categories. These results generate to the setting of symmetric finite tensor categories our results from [5], which were done in the context of commutative algebras over a commutative ring R. In the author's Ph.D. thesis [14] and a paper which emerged out of it together with Stefaan Caenepeel we proceeded the former construction to commutative bialgebroids over R. The idea of the present paper is to bring these constructions to the context of symmetric finite tensor categories and investigate how far we may get in this setting.

We introduce bialgebroid categories over a finite tensor category $\mathcal{C}$. In [16] we introduced $\mathcal{C}$-comonoidal categories as coalgebra objects in the 2-category $(\mathcal{C}\text{- Bimod}, \boxtimes_{\mathcal{C}}, \mathcal{C})$. We define a bialgebroid category $\mathcal{A}$ as a

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$\mathcal{C}$-bimodule category which is monoidal and $\mathcal{C}$-comonoidal with certain compatibility conditions. Intuitively, these conditions require that the commultiplication and the counit functors from the comonoidal structure are monoidal, though some caution is needed, as $\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{A}$ in general is not a monoidal category. To tackle this issue, we present two definitions, one for a non-braided and the other for a braided category $\mathcal{A}$, which obviously coincide in the braided setting. We concentrate on the case where both $\mathcal{C}$ and $\mathcal{A}$ are symmetric as monoidal categories. Then we introduce a new cohomology over $\mathcal{A}$, called Harrison cohomology, which for $\mathcal{A} = \mathcal{C} \boxtimes \mathcal{C}$ coincides with the Amitsur cohomology over $\mathcal{C}$ from [15]. In particular, we have that $H^n(\mathcal{A}, P) = H^{n+1}(\mathcal{C}, P)$, where P is a suitable functor from symmetric tensor categories to abelian groups (or symmetric monoidal categories). Thus n-cocycles over $\mathcal{A}$ are $n+1$-cocycles over $\mathcal{C}$. We record that there exists an analogous infinite exact sequence as in the latter article, which is constructed following *mutatis mutandis* the ideas from therein. In order to interpret the middle cohomology group in the third level of this sequence, we consider quasi-monoidal structures on the bicategory of representations $\mathcal{A}\text{-}\mathrm{Mod}$. These are tensor products on 0-cells with an associativity constraint which satisfies the pentagonal axiom up to a natural equivalence. In general these associativity constraints are not identities and this is why we deal indeed with a bicategory, rather than a 2-category $\mathcal{A}\text{-}\mathrm{Mod}$. We call these structures "quasi-monoidal" as we are not interested at this point in the unit object, nor the corresponding unity constraints and other axioms for a monoidal bicategory structure on $\mathcal{A}\text{-}\mathrm{Mod}$.

We generate quasi-monoidal structures on $\mathcal{A}\text{-}\mathrm{Mod}$ using (additive) autoequivalences of $\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{A}\text{-}\mathrm{Mod}$. For this purpose we develope an Eilenberg–Watts-type Theorem for 2-categories. We do this in two stages. Firstly, in Theorem 3.1 we start with the (2-)category $\mathrm{Pseud}(\mathcal{C}\text{-}\mathrm{Mod}, \mathcal{D}\text{-}\mathrm{Mod})$ of pseudo-functors $\mathcal{C}\text{-}\mathrm{Mod} \longrightarrow \mathcal{D}\text{-}\mathrm{Mod}$ and construct a functor to the (2-)category $\mathcal{D}\text{-}\mathcal{C}\text{-}\mathrm{Bimod}$. We show how the hexagonal coherence diagram for the monoidal structure of a pseudo-functor $\mathcal{F} : \mathcal{C}\text{-}\mathrm{Mod} \longrightarrow \mathcal{D}\text{-}\mathrm{Mod}$ corresponds to the pentagonal axiom for the right $\mathcal{C}$-module category structure of $\mathcal{F}(\mathcal{C})$, while the pentagonal coherence for a pseudo-natural transformation $\omega : \mathcal{F} \longrightarrow \mathcal{G}$ of pseudo-functors $\mathcal{F}, \mathcal{G} : \mathcal{C}\text{-}\mathrm{Mod} \longrightarrow \mathcal{D}\text{-}\mathrm{Mod}$ corresponds to the pentagonal axiom for the right $\mathcal{C}$-linearity of the functor $\omega(\mathcal{C}) : \mathcal{F}(\mathcal{C}) \longrightarrow \mathcal{G}(\mathcal{C})$. The functor $\mathrm{Pseud}(\mathcal{C}\text{-}\mathrm{Mod}, \mathcal{D}\text{-}\mathrm{Mod}) \longrightarrow \mathcal{D}\text{-}\mathcal{C}\text{-}\mathrm{Bimod}$ induced in this way is clearly essentially surjective, though it is not full and fails to be an equivalence of categories. In Theorem 3.4 we restrict to the category $2\text{-}\mathrm{Fun}_{cont}(\mathcal{D}\text{-}\mathrm{Mod}, \mathcal{C}\text{-}\mathrm{Mod})$ of 2-functors which preserve arbitrary coproducts and cokernels and we prove that it is equivalent to $\mathcal{D}\text{-}\mathcal{C}\text{-}\mathrm{Bimod}$. For a consequence, taking $\mathcal{D} = \mathcal{C}$ and considering invertible cells on both levels, we deduce that every autoequivalence of $\mathcal{C}\text{-}\mathrm{Mod}$ is equivalent to $\mathcal{M} \boxtimes_{\mathcal{C}} -$ for some $[\mathcal{M}] \in \mathrm{BrPic}(\mathcal{C})$, where $\mathrm{BrPic}(\mathcal{C})$ denotes the Brauer–Picard group introduced in [11].

Finally, we prove that the quasi-monoidal structures on $\mathcal{A}\text{-}\mathrm{Mod}$ that we study form a monoidal category, whose Grothendieck group is isomorphic to the group $Z^2(\mathcal{A}, \underline{\mathrm{Pic}})$ of 2-cocycles in our Harrison cohomology with values in $\underline{\mathrm{Pic}}(\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{A})$. For $\mathcal{A} = \mathcal{C} \boxtimes \mathcal{C}$ we get an interpretation of the 3-cocycles over $\mathcal{C}$. Given a braided finite tensor category $\mathcal{C}$, we denote by $\underline{\underline{\mathrm{Pic}}}(\mathcal{C})$ the category of invertible one-sided $\mathcal{C}$-bimodule categories. One-sided bimodule categories were studied in [6]. The Grothendieck group $\mathrm{Pic}(\mathcal{C})$ of $\underline{\underline{\mathrm{Pic}}}(\mathcal{C})$ is a subgroup of $\mathrm{BrPic}(\mathcal{C})$. In [15] we proved that when $\mathcal{C}$ is symmetric, the category $(\underline{\underline{\mathrm{Pic}}}(\mathcal{C}), \boxtimes_{\mathcal{C}}, \mathcal{C})$ is symmetric monoidal and thus the group $\mathrm{Pic}(\mathcal{C})$ is abelian. This fact underlies our construction of the Amitsur cohomology over symmetric finite tensor categories in [15] and the Harrison cohomology over symmetric bialgebroid categories in the present paper. Due to [12, Proposition 2.7] we know that any symmetric finite tensor category $\mathcal{C}$ is equivalent to a category of finite-dimensional representations of a finite-dimensional triangular weak quasi-Hopf algebra H. Moreover, by [1, Theorem 5.1.1], [9, Theorem 4.3], if H is a Hopf algebra and the underlying field is algebraically closed and of characteristic zero, H is the Drinfel'd twist of a modified supergroup algebra.

In the coming section we set some preliminary notions and results. Here we deduce a relation between the composition of successive morphisms acting between certain invertible objects in $\mathcal{C}$, on the one hand-side, and their tensor product, on the other. This will be of big importance in Theorem 6.4. Section 3 is dedicated to the Eilenberg–Watts Theorem for 2-categories, Section 4 to the bialgebroid categories and Section 5 to