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Finite groups with only one p-singular Brauer character degree ☆

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ABSTRACT

Recently, Isaacs, Moretó, Navarro, and Tiep investigated finite groups with just one irreducible character degree divisible by a given prime p, and showed that their Sylow p-subgroups are almost normal and almost abelian. In this paper, we consider the corresponding situation for Brauer characters. In particular, we show that if a finite group G has just one irreducible p-Brauer character degree n divisible by $p \geq 5$ then either $G/\mathbf{O}_p(G)$ has a non-normal T.I. Sylow p-subgroup of order n_p, or G has a nonabelian chief factor of order divisible by p that is unique and is a simple group of Lie type of characteristic p.

1. Introduction

Let G be a finite group and let p be a prime. The celebrated Ito–Michler theorem asserts that the group G has a normal abelian Sylow p-subgroup if and only if all of the ordinary irreducible characters of G have p'-degrees, that is, degrees not divisible by p (see [21, Theorem 2.3]). A p-Brauer character analog, due to Michler, says that the group G has a normal Sylow p-subgroup if and only if all of the p-Brauer irreducible characters of G have p'-degrees (see [21, Theorem 2.4]). Relaxing the p'-degree condition, Isaacs, Moretó, Navarro, and Tiep [14] investigated finite groups with just one irreducible character degree divisible by p, and showed that such groups have almost normal and almost abelian Sylow p-subgroups. In this paper, we consider the corresponding situation for Brauer characters.

Theorem 1.1. *Let p be a prime and G a finite p-solvable group with just one irreducible p-Brauer character degree n divisible by p. Suppose that G has an abelian Sylow 2-subgroup if $p = 2$. Then $G/\mathbf{O}_p(G)$ has a non-normal trivial intersection (T.I.) Sylow p-subgroup of order n_p.*

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When $p = 2$ the additional assumption of Theorem 1.1 on Sylow 2-subgroups can not be removed. For example, if $G = S_4$ then the degree set of irreducible 2-Brauer characters of G is $\{1, 2\}$ but the Sylow 2-subgroups of G are dihedral of order 8 and do not intersect trivially.

When considering finite non-p-solvable groups, we first investigate almost simple groups.

Theorem 1.2. *Let H be an almost simple group with socle S of order divisible by p. Then one of the following holds.*

(*a*) *H has two faithful irreducible p-Brauer characters of distinct degrees divisible by p.*
(*b*) *S is a simple group of Lie type of characteristic p.*
(*c*) *$S = L_2(q)$ and $p \mid q^2 - 1$.*
(*d*) *$(S, p) \in \{(J_1, 3), (J_1, 5)\}$.*

In general, we have

Theorem 1.3. *Let p be a prime and G a finite non-p-solvable group with just one irreducible p-Brauer character degree n divisible by p. Then one of the following holds.*

(*a*) *$G/\mathbf{O}_p(G)$ has a non-normal T.I. Sylow p-subgroup of order n_p.*
(*b*) *G has a nonabelian chief factor of order divisible by p that is unique and is a simple group of Lie type of characteristic p, $L_2(q)$ in which case $p = 2$ and q is odd, or $L_2(8)$ in which case $p = 3$.*

We remark here that if G is isomorphic to $U_4(2) \cong S_4(3)$ or the automorphism group of $L_2(8)$, then the degree set of irreducible 3-Brauer characters of G is $\{1, 5, 10, 14, 25, 81\}$ or $\{1, 7, 27\}$, respectively. In both cases, however, the Sylow 3-subgroups of G are not T.I. sets in G. Also, note that if $G = \mathrm{PGL}_2(q)$ with q a Fermat prime then the degree set of irreducible 2-Brauer characters of G is $\{1, q - 1\}$. Thus the simple factors appeared in Theorem 1.3 (b) really occur.

Corollary 1.4. *Let G be a finite group with just one irreducible p-Brauer character degree n divisible by $p \geq 5$. Then either $G/\mathbf{O}_p(G)$ has a non-normal T.I. Sylow p-subgroup of order n_p, or G has a nonabelian chief factor of order divisible by p that is unique and is a simple group of Lie type of characteristic p.*

2. Preliminaries

As usual, let $\mathrm{Irr}(G)$ be the set of ordinary irreducible characters of G, and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ the degree set of $\mathrm{Irr}(G)$. Let $H \leq G$ and $\theta \in \mathrm{Irr}(H)$. Denote by $\mathrm{Irr}(G \mid \theta)$ the set of irreducible constituents of θ^G and by $\mathrm{cd}(G \mid \theta)$ the corresponding degree set. Similarly, let $\mathrm{IBr}(G)$ be the set of irreducible p-Brauer characters of a finite group G, and $\mathrm{cd}_{\mathrm{Br}}(G)$ the degree set of $\mathrm{IBr}(G)$. But we will also use $\mathrm{IBr}_p(G)$ and $\mathrm{cd}_{\mathrm{Br}_p}(G)$ when it is necessary to emphasize the prime p. Let $H \leq G$ and $\theta \in \mathrm{IBr}(H)$. Denote by $\mathrm{IBr}(G \mid \theta)$ the set of irreducible constituents of θ^G and by $\mathrm{cd}(G \mid \theta)$ the corresponding degree set. We will frequently use the results about Brauer characters introduced in Chapter 8 of Navarro's book [22].

As in [14], we similarly say that a finite group G is a *one-p-Brauer-degree group* if $\mathrm{cd}_{\mathrm{Br}}(G)$ has at most one degree divisible by p. We start with two results about characters of finite (almost) simple groups. Then we show some properties of one-p-Brauer-degree groups, the first one of which is about the structure of their minimal normal p'-subgroups.

Lemma 2.1. *Let S be a finite simple group. Then S has a nonlinear* $\mathrm{Aut}(S)$*-extendible ordinary irreducible character.*