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A universal Krull–Lindenbaum theorem ☆

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ABSTRACT

We formulate a natural common generalisation of Krull's theorem on prime ideals and of Lindenbaum's lemma on complete consistent theories; this has instantiations in diverse branches of algebra, such as the Artin–Schreier theorem. Following Scott we put the Krull–Lindenbaum theorem in universal rather than existential form, which move allows us to give a relatively direct proof with Raoult's Open Induction in place of Zorn's Lemma. By reduction to the corresponding theorem on irreducible ideals that is due to Noether, McCoy, Fuchs and Schmidt, we further shed light on why prime ideals occur together with transfinite methods.

1. Introduction

Several indirect proofs with Zorn's Lemma have recently allowed for being turned upside down into direct proofs with Raoult's Open Induction [42,3,8,12], which is transfinite induction limited to Scott-open predicates. Under sufficiently concrete circumstances one may further reduce to induction over finite partial orders, and thus achieve a constructive proof. Then mathematical induction suffices unless one fixes the size of the objects under consideration, in which case one even gets an entirely first-order proof. Case studies pertain to the ideal theory of commutative rings [54] and more specifically to the Gelfand theory of Banach algebras [22].

Toward a systematic treatment we now seek to sort into proof patterns the many instances of indirect proofs with Zorn's Lemma which can be found in mathematical practice. During this undertaking we have come across an extensive generalisation of Krull's theorem [28] on prime ideals and of Lindenbaum's lemma [59] on complete consistent theories. This generalisation subsumes various instances from diverse branches of algebra, such as the Artin–Schreier theorem. Via Lindenbaum's lemma it stands right at the basis of the Gödel completeness theorem for arbitrary languages.

☆ This paper is extracted from Chapter 5 of the first author's doctoral dissertation [46] supervised by the second author.

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Following Scott [55] we put our theorem in universal rather than existential form, in which it is related to what is known as the formal Hilbert Nullstellensatz and more generally to the concept of spatiality in locale theory and formal topology [25,26,10,21,50]. This move moreover allows us to prove the theorem in a relatively direct way, with the aforementioned Open Induction in place of Zorn's Lemma. By reduction to the corresponding theorem on irreducible ideals—due to Noether [37], McCoy [35], Fuchs [20] and Schmidt [52]—we further shed light on what prime ideals and related concepts have to do with transfinite methods.

Although—or just because—the universal Krull–Lindenbaum theorem can rightly be viewed as even more abstract than any of its instances, the availability of a relatively direct and inductive proof is likely to have some impact on the current undertaking [11,32,34,51] of a partial realisation of the revised Hilbert Programme à la Kreisel and Feferman. We expect that, as in the case studies mentioned above, one will eventually be able to do with finite methods and without ideal objects whenever it comes to prove any concrete instantiation whatsoever of the universal theorem; and that just its universal character will suggest a general method.

Throughout this paper, we use the suggestive notation $M \between N$ (rather than $M \cap N \neq \emptyset$) to say that the classes M and N have an element in common[1]; for $M, N \subseteq S$ this can be put as

$$M \between N \;\equiv\; \exists a \in S\,(a \in M \wedge a \in N)\,.$$

Note that $M \between M$ means that M has an element, and that $\{a\} \between N$ means $a \in N$.

2. Preparation

2.1. Open induction

We first recollect—and enrich—some requisites from [54], which includes standard material. Let $(X, \leqslant)$ be a partial order. Every quantification over the variables x, x', y, and z is to be understood as over the elements of X. We sometimes identify a predicate φ on X with $\{x \in X : \varphi(x)\}$.

As usual, $x = y \wedge z$ means that x is the *greatest lower bound*, *infimum* or *meet* of y and z: that is,

$$\forall x' \;(x' \leqslant x \iff x' \leqslant y \;\&\; x' \leqslant z)\;.$$

Likewise, $x = \bigvee Y$ says that x is the *least upper bound*, *supremum* or *join* of $Y \subseteq X$: that is,

$$\forall x' \;(x' \geqslant x \iff \forall y \in Y\, x' \geqslant y)\;.$$

Arbitrary meets and binary joins are dealt with accordingly [56]. Note that it is not required that X always have the meets and joins in question, though this will often be the case.

Let O be a predicate on X. We say that O is *progressive* if

$$\forall x\,(\forall y > x\, O(y) \implies O(x)),$$

where $y > x$ is understood as the conjunction of $y \geqslant x$ and $y \neq x$. By *induction for O on X* we mean the following:

$$\textit{If } O \textit{ is progressive}, \textit{ then } \forall x\, O(x).$$

Note that this is induction from above rather than, as is more common, from below.

[1] We have adopted this notation from Giovanni Sambin.