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Fibrations by non-smooth projective curves of arithmetic genus two in characteristic two

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ABSTRACT

Looking in positive characteristic for failures of the Bertini–Sard theorem, we determine, up to birational equivalence, the separable proper morphisms between smooth algebraic varieties in characteristic two, whose fibres are non-smooth curves of arithmetic genus two. We show that almost all fibres of such a morphism are geometrically elliptic if and only if the function field of the generic fibre is separable over its canonical quadratic rational subfield. We discover a smooth sixfold $Z \subset \mathbb{P}^4 \times \mathbb{A}^5$ such that almost all fibres of the projection morphism $\pi : Z \to \mathbb{A}^5$ are cuspidal geometrically elliptic curves of arithmetic genus two. The main theorem of the paper states that each proper separable morphism between smooth algebraic varieties, whose fibres are geometrically elliptic curves of arithmetic genus two, is birational equivalent to a base extension of the fibration π.

0. Introduction

Bertini's theorem on moving singularities, published in the 1880s, has become a fundamental tool in Algebraic Geometry. Nowadays, due to its similarities to Sard's theorem on differentiable maps, it is also called the Bertini–Sard theorem. It ensures that almost all fibres of a dominant morphism between smooth complex algebraic varieties are smooth.

However, in the 1940s Zariski [24] observed that the theorem may fail in positive characteristic. He had constructed a fibration $\phi : T \to B$ by algebraic curves that admits moving singularities, though the total space T is smooth. A moving singularity of ϕ can be viewed as a horizontal prime divisor on the total space with the property that each of its points is a singular point of the fibre to which it belongs.

Translated into modern language, Zariski argued that, though the generic fibre is a regular scheme over the base field $k(B)$, it may not be smooth, that is, the geometric generic fibre, defined by extending the

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base field to its algebraic closure $\overline{k(B)}$, may have singularities. This means that the function field $k(T)|k(B)$ may be non-conservative, that is, its genus g may decrease by tensoring with $\overline{k(B)}$. For more explications we refer to Section 1.

To rescue Bertini's theorem in positive characteristic p, we are conduced to classify its exceptions. As follows from a theorem of Tate [22], Bertini's theorem can only fail if $p \leq 2g+1$. Non-conservative function fields of genus 1 were classified by Queen [10], and of genus 2 in odd characteristic by Borges Neto [3]. General results on non-conservative function fields and their singular primes were developed by Stichtenoth, Bedoya and the second author in the papers [17,1,18].

In their program to extend Enriques' classification of algebraic surfaces to arbitrary characteristic, Bombieri and Mumford [2] encountered *quasi-elliptic fibrations*, i.e., fibrations by cuspidal curves of arithmetic genus 1. There is a large number of recent papers on classification theory of algebraic varieties and singularities in positive characteristic, too large to put in our references, which can be found by starting the search with [2] and looking successively for citations and references. Singularities of generic fibres in positive characteristic were analyzed by Schröer [15]. Fibrations by non-smooth curves of arithmetic genus 3 in characteristic 3, 5 and 7 were studied by Salomão [13,14] and the second author [20,21].

In the present paper we study in characteristic two the fibrations by non-smooth curves of arithmetic genus two. We realize the fibres by tri-canonical embeddings as curves on a cone in $\mathbb{P}^4$. The discussion naturally divides into two cases. If the function field of the generic fibre is separable over its canonical quadratic rational subfield, then we prove that almost every fibre is geometrically elliptic, i.e., its non-singular model is an elliptic curve (see Theorem 3.1). In the opposite case the fibres are rational, as discussed in Theorem 4.1.

We discover a 6-dimensional smooth algebraic variety $Z \subset \mathbb{P}^4 \times \mathbb{A}^5$ such that almost all fibres of the projection morphism $\pi : Z \to \mathbb{A}^5$ are cuspidal geometrically elliptic curves of arithmetic genus two (see Theorem 5.1). We describe how the elliptic modular invariant of the fibres varies, determine the singular points of all fibres, and discuss how the singularities move.

The main result of this paper is Theorem 5.2. It states that each proper separable morphism between smooth algebraic varieties, whose fibres are geometrically elliptic curves of arithmetic genus two, is birational equivalent to a base extension of the fibration $\pi : Z \to \mathbb{A}^5$. A similar result for fibrations by rational curves of arithmetic genus two is also obtained (see Theorem 5.4).

1. Moving singularities of fibrations by algebraic curves

In this introductory section we present prerequisites on moving singularities of fibrations by algebraic curves, needed to understand our paper.

Let $\phi : T \to B$ be a dominant morphism of irreducible algebraic varieties defined over an algebraically closed field k. We assume that $\dim T = \dim B + 1$ or, equivalently, almost all fibres are algebraic curves (see [16, p. 74]). Thus by restricting if necessary the base variety B to a dense open subvariety we get a fibration by algebraic curves.

By identifying the rational functions on the base B with rational functions on the total space T that are constant along each fibre, we can view the field $k(B)$ of the base as a subfield of $k(T)$. We assume that almost all fibres are integral. By a theorem of Matsusaka this means that $k(B)$ is algebraically closed in $k(T)$ and that $k(T)$ is separable over $k(B)$ (see [9], [16, pp. 139–140]). Thus the field $k(T)$ of the total space, which is a higher dimensional function field over the constant field k, becomes a one-dimensional separable function field over the base field $k(B)$. In this sense, the fibrations by integral algebraic curves over the variety B, up to birational equivalence, correspond bijectively to the isomorphism classes of the one-dimensional separable function fields over $k(B)$.

In the setting of schemes, the function field $k(T)|k(B)$ is the field of the *generic fibre* $\mathcal{T} \times_{\mathcal{B}} \operatorname{Spec} k(B)$, where the calligraphic letters $\mathcal{T}$ and $\mathcal{B}$ stand for the integral schemes whose points correspond bijectively to the closed irreducible subsets of T and B, respectively. The generic fibre is a geometrically integral curve