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Mathieu–Zhao spaces of univariate polynomial rings with non-zero strong radical

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ABSTRACT

We describe all Mathieu–Zhao spaces of the univariate polynomial ring $k[t]$ (k an algebraically closed field of characteristic zero) which have a non-zero strong radical.

0. Introduction

Let k be any field and denote by R a commutative k-algebra. By a subspace V of R we will always mean a k-linear subspace. In [3] the authors introduced the following notions: I_V, the largest ideal of R contained in V, $r(V)$, the *radical of* V, being the set of $a \in R$ such that for some natural number N all elements a^m with $m \geq N$ belong to V and finally $sr(V)$, the *strong radical of* V, being the set of all $a \in R$ such that for each $b \in R$ there exists some natural number N_b such that $ba^m \in V$, for all $m \geq N_b$. One easily verifies that $sr(V) \subseteq r(V)$. The spaces where equality occurs are called *Mathieu–Zhao spaces of* R. Mathieu subspaces were introduced (for more general rings) by Zhao in [9] in order to study several conjectures which all imply the Jacobian Conjecture (see also [1,2,7,8]). The name was changed into Mathieu–Zhao spaces, MZ-spaces, by the first author in [4].

Clearly every ideal of $k[t]$ (the univariate polynomial ring over k) is an MZ-space of it. The ideals of $k[t]$ are easy to describe, they are all principal. On the other hand, describing MZ-spaces of $k[t]$ is still far to complicated. For example the set M of all $f \in \mathbb{C}[t]$ such $\int_0^1 f(t)dt = 0$ is an MZ-space of $\mathbb{C}[t]$ with $r(M) = 0$. Proving this fact is not at all straightforward (see [5]).

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In this paper we make a first step towards a description of the MZ-spaces of $k[t]$. More precisely, we describe all MZ-spaces of $k[t]$ whose strong radical is non-zero, in case k is an algebraically closed field of characteristic zero. This result, the main theorem of this paper, was already obtained in [6].

The proof of the main theorem is based on two ingredients. The first one is Zhao's idempotency theorem (theorem 4.2, [10]) which asserts the following:

Zhao's idempotency theorem. *Let k be a field and A an associative k-algebra, V a k-subspace of A such that all elements of $r(V)$ are algebraic over k. Then V is a (left) MZ-space of A if and only if $Ae \subseteq V$ for every idempotent $e \in V$.*

The second ingredient is a result from the theory of linear recurrence relations. We briefly recall this classical result.

Let k be an algebraically closed field $N \geq 1$ a positive integer and $c_1, \cdots, c_N \in k$, $c_N \neq 0$. An infinite sequence $(a_0, a_1, a_2, \cdots)$ of elements in k is said to *satisfy a recurrence relation defined by the c_i* if

$$a_{N+i} = c_1 a_{N+i-1} + \cdots + c_N a_i, \text{ for all } i \geq 0$$

The set of all sequences $(a_0, a_1, a_2, \cdots)$ satisfying this recurrence relation forms a k-vector space, denoted F, of dimension N. More precisely one can give a basis of F as follows. Consider the *characteristic equation*

$$t^N - c_1 t^{N-1} \cdots - c_{N-1} t - c_N = 0$$

Let Λ be the set of different zeroes of this equation and denote for each $\lambda \in \Lambda$ its multiplicity by $m(\lambda)$. For each $\lambda \in \Lambda$ and each $0 \leq i < m(\lambda)$ the sequence $f_{i,\lambda} := \{n^i \lambda^n\}_{n \geq 0}$ belongs to F. The set of all such sequences forms a k-basis of F. In particular if $a := \{a_n\}_{n \geq 0}$ satisfies the recurrence relation defined by the c_i, then there exist $b_{i,\lambda} \in k$ such that

$$a = \sum_{\lambda \in \Lambda} \sum_{0 \leq i < m(\lambda)} b_{i,\lambda} f_{i,\lambda}$$

So, looking at the n-term in these sequences we get

$$a_n = \sum_{\lambda \in \Lambda} \sum_{0 \leq i < m(\lambda)} b_{i,\lambda} n^i \lambda^n, \text{ for all } n \geq 0$$

1. Preliminaries

From now on V is a k-subspace of $k[t]$, strictly contained in it, and k is an algebraically closed field of characteristic zero. It is shown in [3] that $sr(V) = r(I_V)$. So in particular we get that $sr(V)$ is non-zero if and only if I_V is non-zero. Since we are interested in describing the MZ-spaces with strong radical non-zero, we assume from now on that I_V is *non-zero*. We will write I instead of I_V. Since V is strictly contained in $k[t]$, $I = k[t]f$ for some polynomial f of positive degree. Say

$$f = \prod_{\lambda \in \Lambda} (t - \lambda)^{m(\lambda)}$$

where Λ denotes the set of different zeroes of f in k and $m(\lambda)$ denotes the multiplicity of λ. We may assume that Λ does not contain 0 (just replace t by $t - c$ for some suitable $c \in k$ and observe that sending t to $t - c$ is a k-automorphism of $k[t]$). Since I is contained in V, it is shown in [10] that V is an MZ-pace of $k[t]$ if and only if $M := V/I$ is an MZ-space of the ring $A := k[t]/I$. So we need to investigate MZ-spaces of A.