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Minimal fields of definition for Galois action

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ABSTRACT

Let K be a field, and let $f : X \to Y$ be a finite étale cover between reduced and geometrically irreducible K-schemes of finite type such that f_{K_s} is Galois. Assuming f admits a Galois K-form $\bar{f} : \bar{X} \to Y$, we use it to analyze fields of definition over K for the Galois property of f and the presence of K-points in general K-forms $f' : X' \to Y$ over $Y(K)$.

Additionally, we show that if K is Hilbertian, the group G is non-abelian, and the base variety is rational, then there are finite separable extensions L/K such that some L-form of f_L does not descend to a cover of Y.

1. Introduction

The focus of this paper is the descent of G-Galois covers. For a finite group G, a map of varieties (over a fixed field) is said to be a *G-Galois cover* if it is finite, étale, and G acts freely and transitively on its geometric fibers. (See Definition 2.1.)

David Harbater and Kevin Coombes have made several observations in [1] about the Galois property of descents. Let K be a number field, let Y be a reduced geometrically irreducible K-scheme of finite type, fix a Galois finite étale surjection over Y_{K_s} with Galois group G, and let M be its associated "field of moduli". Under the additional assumption that $Y(K) \neq \emptyset$, [1, Proposition 2.5] shows that there is a descent to a cover of Y_M which is possibly not Galois; and [1, Proposition 2.7] shows that M is the intersection of the number fields F for which there is a descent to a *Galois* cover of Y_F.

The setup of this paper will be slightly different. Let K be any field, and let $f : X \to Y$ be a finite étale surjection between reduced and geometrically irreducible K-schemes of finite type such that f_{K_s} is Galois with Galois group G. It is easy to show (see the beginning of Section 2) that there exists a unique minimal subfield $E \subset K_s$ over K so that $X_E \to Y_E$ is Galois. (Henceforth, this is called the *minimal field of Galois action* of $X \to Y$.) The main theorems (Theorems 2.2, 2.3 and 2.5), shed light on how E is determined, under the assumption that there exists *some* Galois K-form $\bar{X} \to Y$.

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In Proposition 4.1 we show that if K is Hilbertian, the group G is non-abelian, and the base variety is rational, then there exist finite separable extensions L/K such that some L-form of f_L does not descend to a cover of Y.

2. Main theorems

In this section, we state our main results, proved at the end of Section 3.

Definition 2.1. A map $f : X \to Y$ between integral noetherian schemes is called a *cover* if it is a finite, étale surjection. Letting $G := \mathrm{Aut}(X/Y)^{\mathrm{opp}}$, the fiber-degree is a constant $d > 0$ with $|G| \leq d$, and $|G| = d$ if and only if X is a right G-torsor over Y. In such cases we say that f is *Galois* (and G is naturally identified with the Galois group of the extension of function fields). For a finite group Γ, we say that f is a Γ-*cover* if it is Galois and an isomorphism $\Gamma \cong \mathrm{Aut}(X/Y)^{\mathrm{opp}}$ is specified. The notion of *isomorphism* for covers and Γ-covers is defined in the obvious manner.

In what follows, we fix a reduced and geometrically irreducible K-scheme Y of finite type and a cover $f : X \to Y$ such that X is geometrically irreducible over K and f_{K_s} is Galois.

Since $X \to Y$ is étale, the sheaf $\underline{\mathrm{Aut}}_{X/Y}$ is representable by a group scheme over Y (also denoted $\underline{\mathrm{Aut}}_{X/Y}$). Clearly, $X \to Y$ is a left $\underline{\mathrm{Aut}}_{X/Y}$-torsor, and $(\underline{\mathrm{Aut}}_{X/Y})_{Y_{K_s}} = \underline{\mathrm{Aut}}_{X_{K_s}/Y_{K_s}} \cong G^{\mathrm{opp}}$ is a constant Y_{K_s}-group. Let the continuous homomorphism

$$\rho : \mathrm{Gal}(K_s/K) = \mathrm{Gal}(K_s(Y)/K(Y)) \to \mathrm{Aut}(G^{\mathrm{opp}})$$

be the Galois action induced by $\underline{\mathrm{Aut}}_{X/Y}$ as a Y_{K_s}/Y-form of G^{opp}. Then, clearly, the splitting field $E := (K_s)^{\ker(\rho)}$ of ρ is the unique minimal subfield of K_s containing K for which the base change $X_E \to Y_E$ is Galois. We remark that E/K is Galois, and its Galois group is canonically isomorphic to a subgroup of $\mathrm{Aut}(G^{\mathrm{opp}})$.

Theorem 2.2. *With the notation as above, assume that there exists* **some** *Galois K-form $\bar{X} \to Y$ of f, and let P be a point such that $X_P(K) \neq \emptyset$. Let T be the P-fiber $\bar{X}_P$ viewed as a right G-torsor over* $\mathrm{Spec}(K)$. *Then the minimal field of Galois action for $X \to Y$ is contained in the splitting field of T.*

We may conclude from Theorem 2.2 that the minimal field of Galois action for $X \to Y$ is *contained in the specialization at P of all Galois K-forms of f*. The effect of changing the point P in Theorem 2.2 is expressed by the following result:

Theorem 2.3. *With the notation of Theorem 2.2, let Q be another point in $Y(K)$. Then the fiber over Q in $X \to Y$ has a K-rational point if and only if $\bar{X}_P$ and $\bar{X}_Q$ are isomorphic as right G-torsors.*

Remark 2.4. A variant of Theorem 2.3 has been known before, and goes by the name the "Twisting Lemma" ([8, 2.2]; [4, Section 2]; see also Lemma 3.3). It has been applied in Galois-Theoretic contexts, most notably by Pierre Dèbes [2–6]. However, the Twisting Lemma is a bit weaker since it merely says that if f is Galois then it admits *some* K-form $X' \to Y$ such that K-points in fibers over $Y(K)$ can be detected by fibers of f as in Theorem 2.3.

Finally, I will prove the following.