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The full exceptional collections of categorical resolutions of curves

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ABSTRACT

This paper gives a complete answer of the following question: which (singular, projective) curves have a categorical resolution of singularities which admits a full exceptional collection? We prove that such full exceptional collection exists if and only if the geometric genus of the curve equals to 0. Moreover we can also prove that a curve with geometric genus equal or greater than 1 cannot have a categorical resolution of singularities which has a tilting object. The proofs of both results are given by a careful study of the Grothendieck group and the Picard group of that curve.

1. Introduction

For a triangulated category $\mathcal{C}$, having a full exceptional collection is a very good property. Recall that the definition of full exceptional collection is as follows.

Definition 1.1. A full exceptional collection of a triangulated category $\mathcal{C}$ is a collection $\{A_1 \dots A_n\}$ of objects such that

(1) for all i one has $\mathrm{Hom}_{\mathcal{C}}(A_i, A_i) = k$ and $\mathrm{Hom}_{\mathcal{C}}(A_i, A_i[l]) = 0$ for all $l \neq 0$;
(2) for all $1 \leq i < j \leq n$ one has $\mathrm{Hom}_{\mathcal{C}}(A_j, A_i[l]) = 0$ for all $l \in \mathbb{Z}$;
(3) the smallest triangulated subcategory of $\mathcal{C}$ containing $A_1, \dots, A_n$ coincides with $\mathcal{C}$.

However it is not very common that a triangulated category $\mathcal{C}$ has a full exceptional collection. In algebraic geometry, it is well-known that for a smooth projective curve X over an algebraically closed

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field k, its bounded derived category of coherent sheaves $\mathrm{D}^b(\mathrm{coh}(X))$ has a full exceptional collection if and only if the genus of X equals to 0.

Moreover for a singular projective curve X and a (geometric) resolution of singularities $\widetilde{X} \to X$, the geometric genus of $\widetilde{X}$ and X are equal, hence it is clear that $\mathrm{D}^b(\mathrm{coh}(\widetilde{X}))$ has a full exceptional collection if and only if the geometric genus of X equals to 0.

In this paper we would like to consider the categorical resolution of X, which is introduced in [4].

Definition 1.2. (See [4] Definition 3.2 or [5] Definition 1.3.) A categorical resolution of a scheme X is a smooth, cocomplete, compactly generated, triangulated category $\mathscr{T}$ with an adjoint pair of triangulated functors

$$\pi^* : \mathrm{D}(X) \to \mathscr{T} \text{ and } \pi_* : \mathscr{T} \to \mathrm{D}(X)$$

such that

(1) $\pi_* \circ \pi^* = id$;
(2) both π_* and π^* commute with arbitrary direct sums;
(3) $\pi_*(\mathscr{T}^c) \subset \mathrm{D}^b(\mathrm{coh}(X))$ where $\mathscr{T}^c$ denotes the full subcategory of $\mathscr{T}$ which consists of compact objects.

Remark 1. The first property implies that π^* is fully faithful and the second property implies that $\pi^*(\mathrm{D}^{\mathrm{perf}}(X)) \subset \mathscr{T}^c$.

Remark 2. The categorical resolution of X is not necessarily unique.

Remark 3. In this paper we will not discuss further on the smoothness of a triangulated category and the interested readers may refer to [5] Section 1. Moreover, the main result in this paper does not depend on the smoothness, see Corollaries 3.6 and 4.8 below.

We are interested in the question that when does $\mathscr{T}^c$ have a full exceptional collection. If X is an projective curve of geometric genus $g = 0$, it can be deduced from the construction in [5] that there exists a categorical resolution $(\mathscr{T}, \pi^*, \pi_*)$ of X such that $\mathscr{T}^c$ has a full exceptional collection. See Proposition 4.1 below.

The main result of this paper is the following theorem, which rules out the possibility for any categorical resolution of a curve with geometric genus $g \geq 1$ has a full exceptional collection.

Theorem 1.1. *(See Theorem 4.9 below.) Let X be a projective curve over an algebraically closed field k. Let $(\mathscr{T}, \pi^*, \pi_*)$ be a categorical resolution of X. If the geometric genus of X is ≥ 1, then $\mathscr{T}^c$ cannot have a full exceptional collection.*

In other words, X has a categorical resolution which admits a full exceptional collection if and only if the geometric genus of X equals to 0.

Remark 4. In a recent paper [1] a result which is related to the above claim has been proved. Actually it has been proved that if X is a reduced rational curve, then there exists a categorical resolution $(\mathscr{T}, \pi^*, \pi_*)$ of X such that $\mathscr{T}^c$ has a tilting object, which in general does not come from an exceptional collection. See [1] Theorem 7.4.

Recall that the definition of tilting object is given as follows.

Definition 1.3. Let $\mathcal{C}$ be a triangulated category. A tilting object is an object L of $\mathcal{C}$ which satisfies the following properties.