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On parameterized differential Galois extensions

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ABSTRACT

We prove some existence results on parameterized strongly normal extensions for logarithmic equations. We generalize a result in Wibmer (2012) [16]. We also consider an extension of the results in Kamensky and Pillay (2014) [4] from the ODE case to the parameterized PDE case. More precisely, we show that if $\mathcal{D}$ and Δ are two distinguished sets of derivations and $(K^{\mathcal{D}}, \Delta)$ is existentially closed in (K, Δ), where K is a $\mathcal{D} \cup \Delta$-field of characteristic zero, then every (parameterized) logarithmic equation over K has a parameterized strongly normal extension.

1. Introduction

Let $\Pi = \mathcal{D} \cup \Delta = \{D_1, \dots, D_r\} \cup \{\delta_1, \dots, \delta_{m-r}\}$ be a set of commuting derivations, with $m \geq r > 0$, and K be a Π-field of characteristic zero. Consider the (parameterized) system of homogeneous linear differential equations

$$D_1 Y = A_1 Y, \ \dots, \ D_r Y = A_r Y, \quad \text{with } Y \text{ ranging in } \mathrm{GL}_n, \tag{$\star$}$$

where the A_i's are $n \times n$ matrices with entries from the differential field K satisfying the usual integrability condition

$$D_i A_j - D_j A_i = [A_i, A_j], \quad \text{for } i, j = 1, \dots, r.$$

Recall that a parameterized Picard–Vessiot (PPV) extension of K for $(\star)$ is a Π-field extension L of K such that

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1. L is generated over K by the entries (and all Π-derivatives) of a matrix solution $Z \in \mathrm{GL}_n(L)$ of $(\star)$; in other words, $L = K\langle Z\rangle_\Pi = K\langle Z\rangle_\Delta$, and
2. $L^{\mathcal{D}} = K^{\mathcal{D}}$, that is the field of $\mathcal{D}$-constants of L is the same as the field of $\mathcal{D}$-constants of K.

Above, and henceforth, we use the notation $K\langle Z\rangle_\Pi$ for the Π-field generated by Z over K, and $K\langle Z\rangle_\Delta$ for the Δ-field generated by Z over K.

For an example, take $K = (\mathbb{C}(x,t), \{\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\})$, where we think of $\mathcal{D}$ as $\{\frac{\partial}{\partial x}\}$ and the parametric derivations Δ as $\{\frac{\partial}{\partial t}\}$. Let $(\star)$ be

$$\frac{\partial y}{\partial x} = \frac{t}{x} y. \tag{1}$$

Clearly, $y = x^t$ is a solution and as $\frac{\partial}{\partial t}(x^t) = x^t \cdot \log x$, one is interested in the field $L = K(x^t, \log x)$. It turns out (see [2, Example 3.1]) that L is indeed a PPV extension of K for equation (1).

Parameterized Picard–Vessiot extensions were introduced in [2] by Cassidy and Singer as a fundamental tool for studying parametric equations such as equation $(\star)$. In particular, they capture valuable information about the algebraic relations that exist among a set of solutions as well as their Δ-algebraic relations (where Δ is the set of parametric derivations). PPV extensions have attracted much attention in recent years and we direct the reader to [10] for some applications of the parameterized Picard–Vessiot theory. It should be noted that PPV extensions do not always exist; for example, consider the differential field $(\mathbb{R}(t), D, \frac{d}{dt})$ where D is the trivial derivation. Adjoin to $\mathbb{R}(t)$ the general solution b of the system

$$(Dy)^2 + 4y^2 + 1 = 0 \quad \text{and} \quad \frac{dy}{dt} = 0.$$

Let $K = \mathbb{R}(t)\langle b\rangle_\Pi = \mathbb{R}(t, b, Db)$. Note that K is a Π-field where we are thinking of $\mathcal{D} = \{D\}$ and a parametric derivation $\Delta = \{\frac{d}{dt}\}$. Note that if the $\mathcal{D}$-constants $K^{\mathcal{D}}$ were not algebraic over $\mathbb{R}(t)$ then K would be algebraic over $K^{\mathcal{D}}$, but a field of constants is always relatively algebraically closed and so we would get that $K = K^{\mathcal{D}}$ which is impossible as $Db \neq 0$. Thus, $K^{\mathcal{D}}$ is algebraic over $\mathbb{R}(t)$. On the other hand, the polynomial $z^2 + 4x^2 + 1$ is absolutely irreducible, so $\mathbb{R}(t)$ is relatively algebraically closed in $\mathbb{R}(t, b, Db) = K$ (see [13, Theorem 10]). Hence, $K^{\mathcal{D}} = \mathbb{R}(t)$. Now, take $(\star)$ to be the linear differential equation

$$D^2 y + y = 0.$$

The same argument as in [12, §6] shows that if f is a nontrivial solution of this equation, then either

$$\frac{bf^2 + (Db)(Df)f - b(Df)^2}{f^2 + (Df)^2}$$

is a $\mathcal{D}$-constant which is not in $\mathbb{R}(t)$, or $i \in K\langle f\rangle_\Pi$ (where $i^2 = -1$). In both cases we obtain a $\mathcal{D}$-constant of $K\langle f\rangle_\Pi$ that is not in $K^{\mathcal{D}}$. Thus, there are no PPV extensions of K for this equation.

It has been known for quite some time that to get a general existence result, one needs to impose additional assumptions on $K^{\mathcal{D}}$. In [2], Cassidy and Singer showed that if $(K^{\mathcal{D}}, \Delta)$ is Δ-closed, then the existence of a PPV extension of K is guaranteed. This was later improved in [3] by Gillet et al. where they show that the assumption can be weaken to $(K^{\mathcal{D}}, \Delta)$ being existentially closed in (K, Δ); that is, every Δ-algebraic variety over $K^{\mathcal{D}}$ with a K-point has also a $K^{\mathcal{D}}$-point. Examples of the latter occur for instance when K is formally real and $(K^{\mathcal{D}}, \Delta)$ is a real closed ordered differential field; more precisely, $(K^{\mathcal{D}}, \Delta)$ is a model of $RCF \cup UC_{m-r}$ (the theory of real closed ordered differential fields with $m - r$ commuting derivations) introduced by Tressl in [15, §8] (in the case $m - r = 1$ this theory was introduced by Singer [14]). A similar