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Kummer spaces in symbol algebras of prime degree

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ABSTRACT

We classify the monomial Kummer subspaces of division symbol algebras of prime degree p, showing that every such space is standard, and in particular the dimension is no greater than $p+1$. It follows that in a generic symbol algebra, the dimension of any Kummer subspace is at most $p+1$.

1. Introduction

Given an integer n and a central simple F-algebra A whose degree is a multiple of n, an n-Kummer element is an element $v \in A$ satisfying $v^n \in F^\times$ and $v^{n'} \notin F$ for any $1 \leq n' < n$. (We omit n when it is obvious from the context.) These elements play an important role in the structure and presentations of these algebras. For example, in case $\deg(A) = n$ and F is a field of characteristic prime to n containing a primitive nth root of unity, A is cyclic if and only if it contains a Kummer element. Without roots of unity, this equivalence holds when n is prime, but there are counterexamples for general n. (See [7].)

A Kummer subspace of A is an F-vector subspace V where every $v \in V \setminus \{0\}$ is Kummer. In case F is of characteristic prime to n containing a primitive nth root of unity ρ, every cyclic algebra of degree n over F can be presented as a symbol algebra, that is

$$(\alpha, \beta)_{n,F} = F[x, y \,|\, x^n = \alpha,\, y^n = \beta,\, yxy^{-1} = \rho x]$$

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for some $\alpha, \beta \in F^\times$. Assume A is a tensor product of m symbol algebras of degree n over F, and fix a presentation

$$A = \bigotimes_{k=1}^{m} (\alpha_k, \beta_k)_{n,F} = \bigotimes_{k=1}^{m} F[x_k, y_k : x_k^n = \alpha_k, y_k^n = \beta_k, y_k x_k y_k^{-1} = \rho x_k].$$

Definition 1.1. A ***monomial Kummer subspace*** of A (with respect to that fixed presentation) is a Kummer space spanned by elements of the form $\prod_{k=1}^{m} x_k^{a_k} y_k^{b_k}$ for some $0 \le a_1, b_1, \ldots, a_m, b_m \le n-1$.

Assume from now on that $n = p$ is prime. In [6], the author made use of the existence of $(mp+1)$-dimensional monomial Kummer spaces in A to prove that the symbol length of any central simple F-algebras is bounded from above by $p^{r-1} - 1$ when F is a C_r field. We are interested therefore in the maximal possible dimension of Kummer spaces in general, and monomial Kummer spaces in particular. Another motivation comes from the generalized Clifford algebras: if $p+1$ is indeed the maximal dimension of a Kummer space in a symbol algebra of degree p, as we conjecture, then the Clifford algebra of a nondegenerate homogeneous polynomial form of degree p in more than $p+1$ variables cannot have simple images of degree p. (See [2] for more information on generalized Clifford algebras.)

In tensor products of m quaternion algebras, the dimension of Kummer spaces is bounded by $2m+1$. This is an immediate result of the theory of Clifford algebras of quadratic forms. (See [4] for further information.) The Kummer subspaces of cyclic algebras of degree 3 were classified in [10], and then in [8] and [9], using techniques of composition algebras suggested by J.-P. Tignol. The monomial Kummer subspaces of the tensor product of m symbol algebras of degree 3 were classified in [1], establishing an upper bound of $3m+1$. This upper bound holds also for non-monomial Kummer spaces in the generic tensor product of m symbol algebras.

In this paper we study Kummer subspaces in symbol algebras of degree p for any prime p. We prove that the dimension of monomial Kummer spaces in such algebras is bounded by $p+1$. The proof of this algebraic fact requires a nontrivial result from elementary number theory ([5], see also [3]). Finally, we prove in Section 4 that $p+1$ is the upper bound for the dimension of any Kummer subspace in the generic symbol algebra.

2. Kummer subspaces

Let p be a prime number, F be a field of characteristic either 0 or greater than p containing a primitive pth root of unity ρ, and A be a symbol division algebra of degree p over F. The variety X_A of all Kummer elements in A is defined by the condition $s_1 = \cdots = s_{p-1} = 0$, where s_i are the generic characteristic coefficients. We assume that $p \ge 5$.

2.1. Standard Kummer subspaces

Let $x \in X_A$ and let $\bar{x}$ denote its image in $A^\times/F^\times$. Then $\bar{x}$ acts on A by conjugation and its action is of order p, by definition. As we assume $\rho \in F$ we get a decomposition of A:

$$A = V_0(x) \oplus \cdots \oplus V_{p-1}(x)$$

where $V_k(x) = \{w \in A :\ wx = \rho^k xw\}$.

Remark 2.1. It is not hard to see that:

1. for $1 \le k \le p-1$, $V_k(x) = F[x]y^k$ for any fixed nonzero $y \in V_1(x)$.
2. $V_i(x)V_j(x) = V_{i+j \pmod p}$.