Journal of Pure and Applied Algebra 220 (2016) 3437–3449

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Extensions of tangent cones of monomial curves

I-Chiau Huang

Institute of Mathematics, Academia Sinica, 6F, Astronomy–Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, Taiwan, ROC

ARTICLE INFO

Article history:
Received 27 August 2015
Received in revised form 10 March 2016
Available online 25 April 2016
Communicated by S. Iyengar

MSC:
13A30; 13H10

ABSTRACT

Characterizations for the Cohen–Macaulayness, Gorensteiness and complete intersection of extensions of tangent cones of monomial curves are given in terms of basis monomials. As applications, we are able to construct new tangent cones within prescribed singularities from given ones.

1. Introduction

Throughout this paper, κ is a field. For an increasing sequence $t_0, t_1, \cdots, t_n$ of positive integers, the locus $\{(a^{t_0}, \cdots, a^{t_n}) \colon a \in \kappa\}$ in the affine $n+1$ space is called a monomial curve. In the literature, it is often assumed that $t_0, \cdots, t_n$ are relatively prime. Under this assumption, the monoid generated by $t_0, \cdots, t_n$ is called a *numerical semigroup* and is characterized as a subset of $\mathbb{N}$, which is closed under addition, contains 0, and has a finite complement in $\mathbb{N}$. In this paper, the tangent cone of a monomial curve always refers to the tangent cone at the origin. It is also simply called a tangent cone. Our goal is to study such tangent cones from a relative point of view. Topics of interest are singularities including Cohen–Macaulayness, Gorensteiness and complete intersection.

In order to construct explicitly a residual complex of a tangent cone and to read the information of the tangent cone from the complex [9], we were led to work with the monoid generated by the rational numbers $t_0/t_0, \cdots, t_n/t_0$. The assumption on $t_0, \cdots, t_n$ to be relatively prime is irrelevant in the setting of rational numbers. In this paper, we coin a term for such monoids of rational numbers. Recall that a positive integer t is called a *common denominator* of a set S of rational numbers if $tS \subset \mathbb{Z}$.

Definition 1.1 *(Normalized numerical semigroup).* A *normalized numerical semigroup* is a set of rational numbers with a common denominator, which is closed under addition and contains 0, 1 and other numbers greater than 1.

E-mail address: ichuang@math.sinica.edu.tw.

http://dx.doi.org/10.1016/j.jpaa.2016.04.008
0022-4049/

The smallest positive number in a numerical semigroup is called the multiplicity of the semigroup. A numerical semigroup divided by its multiplicity is a normalized numerical semigroup. A normalized numerical semigroup multiplied by its least common denominator is a numerical semigroup. Therefore a normalized numerical semigroup is nothing but a numerical semigroup scaled so that the smallest nonzero element is 1.

In this paper, abusing the notation, we do not distinguish an affine variety from its coordinate ring. So an affine line is a polynomial ring of one variable. As in the literature, a monomial curve is also a numerical semigroup ring; the tangent cone of a monomial curve at the origin is the associated graded ring of a numerical semigroup ring. The monomial curve defined by $t_0, \cdots, t_n$ and its tangent cone are both exponential counterparts of the normalized numerical semigroup S generated by $t_0/t_0, \cdots, t_n/t_0$. For these two counterparts, let $\kappa[\mathbf{e}^S]$ be the monomial curve and $\kappa[X^S]$ be its tangent cone. The former is an integral domain. The structure of the latter is determined by vanishing of products of monomials. As an advantage of our new notation, if S contains another normalized numerical semigroup S', we obtain immediately a homomorphism $\kappa[X^{S'}] \to \kappa[X^S]$ of graded κ-algebras. There are two logarithms in our context. Dropping bases $\mathbf{e}$ and X from the notation of logarithm, we have $\log \mathbf{e}^s = s$ and $\log X^s = s$ for $s \in S$. In the literature, a numerical semigroup ring also refers to the completion of the local ring $\kappa[\mathbf{e}^S]$ at the origin, which is denoted by $\kappa[[\mathbf{e}^S]]$.

The study of singularities of the tangent cone $\kappa[X^S]$ caught attention in recent years. There are characterizations for complete intersection [5], Gorensteiness [3], Cohen–Macaulayness [2,7] and Buchsbaumness [4] mostly in terms of Apéry sets. Recall that the Apéry set of the numerical semigroup t_0S consists of all elements t_0s such that $\mathbf{e}^s$ is not divisible by $\mathbf{e}$. Our characterizations for singularities of a tangent cone are given in terms of another exponential analogue of the Apéry set, namely, the set of basis monomials. In the classical case (also called the absolute case to be made precise in the next paragraph), a basis monomial is an element X^s in $\kappa[X^S]$ which is not divisible by X. Note that, for a Cohen–Macaulay tangent cone, the Apéry set scaled by the multiplicity consists exactly the logarithm of basis monomials. While characterizations for complete intersection [5] and Gorensteiness [3] are for tangent cones that are Cohen–Macaulay or satisfy other conditions, our generalization in terms of basis monomials works for general tangent cones. See [9, Theorem 5.2] for Gorensteiness. See Theorem 5.5 (and its remark) of the current paper for complete intersection.

Following Grothendieck's philosophy on algebraic geometry and commutative algebra, we emphasize morphisms rather than varieties or rings. So the focus of our study is on a tangent cone over another. Nevertheless, as a Noether normalization, the tangent cone of a monomial curve always contains an affine line. Therefore a tangent cone can be studied as a tangent cone over an affine line, and thus fits in our framework. The situation for a tangent cone over an affine line is called the absolute case, versus the relative case referring to the general situation. In this paper, we show that the composition of extensions of tangent cones preserves Cohen–Macaulayness, Gorensteiness and complete intersection. Applied to the absolute case, we are able to construct new tangent cones from given ones within prescribed singularities.

In the literature, the scenario of constructions of tangent cones is often set up by gluings of numerical semigroups [1,10]. Let t_1 and t_2 be the least common denominators of normalized numerical semigroups S_1 and S_2, respectively. A *gluing* of $\kappa[[\mathbf{e}^{S_1}]]$ and $\kappa[[\mathbf{e}^{S_2}]]$ is a numerical semigroup ring $\kappa[[\mathbf{e}^S]]$, where S multiplied by its least common denominator is generated by $p_1t_1S_1$ and $p_2t_2S_2$ for relatively prime integers $p_1 \in t_2S_2$ and $p_2 \in t_1S_1$ such that p_1 is not a part of minimal generators of t_2S_2 and p_2 is not a part of minimal generators of t_1S_1. Numerical semigroup rings behave well under gluings. If $\kappa[[\mathbf{e}^{S_1}]]$ and $\kappa[[\mathbf{e}^{S_2}]]$ are Gorenstein or complete intersection, so are their gluings [6,12]. This is not true for their tangent cones. In general $\kappa[X^S]$ may be not Cohen–Macaulay, even if $\kappa[X^{S_1}]$ and $\kappa[X^{S_2}]$ are Cohen–Macaulay [1]. To tailor the construction, nice gluings [1] and specific gluings [10] were introduced.

Our approach to tangent cones provides another perspective of gluings. Observe that $\kappa[[\mathbf{e}^{S_1}]]$ and $\kappa[[\mathbf{e}^{S_2}]]$ play the same role for their gluing $\kappa[[\mathbf{e}^S]]$. This is not the case for tangent cones. For the gluing $\kappa[X^S]$, one of