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Generalized dimension subgroups and derived functors

Roman Mikhailov [a], Inder Bir S. Passi [b,c,*]

[a] St Petersburg Department of Steklov Mathematical Institute and Chebyshev Laboratory, St Petersburg State University, 14th Line, 29b, Saint Petersburg 199178, Russia
[b] Centre for Advanced Study in Mathematics, Panjab University, Sector 14, Chandigarh 160014, India
[c] Indian Institute of Science Education and Research, Mohali (Punjab) 140306, India

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ABSTRACT

Every two-sided ideal $\mathfrak{a}$ in the integral group ring $\mathbb{Z}[G]$ of a group G determines a normal subgroup $G \cap (1 + \mathfrak{a})$ of G. In this paper certain problems related to the identification of such subgroups, and their relationship with derived functors in the sense of Dold–Puppe, are discussed.

0. Introduction

Let G be a group and $\mathbb{Z}[G]$ its integral group ring. Every two-sided ideal $\mathfrak{a}$ in the integral group ring $\mathbb{Z}[G]$ of a group G determines a normal subgroup $G\cap(1+\mathfrak{a})$ of G. The identification of such normal subgroups is a fundamental problem in the theory of group rings. On expressing the group G as a quotient of a free group F, the problem can clearly be viewed as an identification problem in the free group ring $\mathbb{Z}[F]$. Let $\mathfrak{g}$ denote the augmentation ideal of $\mathbb{Z}[G]$. In analogy with the case when $\mathfrak{a}$ is a power $\mathfrak{g}^n$, $n \geq 1$, of the augmentation ideal and the corresponding subgroup, denoted $D_n(G)$, is called the nth *dimension subgroup* of G, we call the normal subgroups $G \cap (1 + \mathfrak{a} + \mathfrak{g}^n)$, $n \geq 1$, *generalized dimension subgroups*. For every group G and integer $n \geq 1$, it is easily seen that $D_n(G) \geq \gamma_n(G)$, the nth term in the lower central series of G. It is well-known that $D_3(G) = \gamma_3(G)$ (see [15], [7], p. 75), or, equivalently, that $F \cap (1 + \mathfrak{r}\mathbb{Z}[F] + \mathfrak{f}^3) = R\gamma_3(F)$ for every free presentation $G \cong F/R$. A relationship between generalized dimension subgroups and derived functors in the sense of Dold–Puppe [4] is noticed on considering the subgroup $F \cap (1 + \mathfrak{r}\mathfrak{f} + \mathfrak{f}^3)$ (Theorem 12; [8], Theorem 3.3). Motivated by this observation, our aim in this paper is to explore further relations between

* Corresponding author at: Centre for Advanced Study in Mathematics, Panjab University, Sector 14, Chandigarh 160014, India.
E-mail addresses: romanvm@mi.ras.ru (R. Mikhailov), ibspassi@yahoo.co.in (I.B.S. Passi).

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generalized dimension subgroups and derived functors. We expect our approach will initiate further work leading to deeper understanding of subgroups determined by ideals in group rings; for instance, see [14].

Based on the investigations of Narain Gupta [7], we identify, in Section 1, generalized dimension subgroups $F \cap (1+\mathfrak{a}+\mathfrak{f}^4)$ for various two-sided ideals $\mathfrak{a}$ in a free group ring $\mathbb{Z}[F]$. In view of the fact that the dimension series $\{D_n(G)\}_{n\geq 1}$ and the lower central series $\{\gamma_n(G)\}_{n\geq 1}$, in general, cease to be identical from $n = 4$ onwards, the identification of these subgroups becomes of particular interest. Typical of the cases that we consider is the identification of the subgroup $F \cap (1 + \mathfrak{f}\mathfrak{r}\mathfrak{f} + \mathfrak{f}^4)$ (Theorem 5). In Corollary 10 we show that $F \cap (1 + \mathfrak{r}^2\mathbb{Z}[F] + \mathfrak{f}^4) \leq R\gamma_4(F)$, thus answering Narain Gupta's Problem 6.9(a) in [7]. This result, however, leaves open for future investigation the interesting question whether $F \cap (1 + \mathfrak{r}^{n-1}\mathbb{Z}[F] + \mathfrak{f}^{n+1}) \leq R\gamma_{n+1}(F)$ for some $n \geq 4$.

In Section 2 we exhibit how some of the generalized dimension subgroups are related to derived functors. Our main result here is Theorem 13 relating $F \cap (1 + \mathfrak{f}\mathfrak{r}\mathfrak{f} + \mathfrak{f}^4)$ to the homology of the Eilenberg–MacLane space $K(G_{ab}, 2)$, where G_{ab} is the abelianization of the group $G \cong F/R$. More precisely, we prove that, if G_{ab} is 2-torsion-free, then $\frac{F\cap(1+\mathfrak{f}\mathfrak{r}\mathfrak{f}+\mathfrak{f}^4)}{[R,\, R,\, F]\gamma_4(F)} \cong L_1\,\mathrm{SP}^3(G_{ab})$, where $L_1\,\mathrm{SP}^3$ is the first derived functor of the third symmetric power functor, and there is a natural short exact sequence

$$\frac{F \cap (1 + \mathfrak{f}\mathfrak{r}\mathfrak{f} + \mathfrak{f}^4)}{[R,\, R,\, F]\gamma_4(F)} \hookrightarrow H_7K(G_{ab},\, 2) \twoheadrightarrow \mathrm{Tor}(G_{ab},\, \mathbb{Z}/3\mathbb{Z}). \tag{1}$$

Consequently, if G_{ab} is p-torsion-free for $p = 2,\ 3$, then there is a natural isomorphism

$$\frac{F \cap (1 + \mathfrak{f}\mathfrak{r}\mathfrak{f} + \mathfrak{f}^4)}{[R,\, R,\, F]\gamma_4(F)} \cong H_7K(G_{ab},\, 2). \tag{2}$$

We define a functor $\mathfrak{L}^3_s(A)$ on the category $\mathfrak{A}$ of abelian groups (see Section 2 for definition) and prove (Theorem 15) that if G is a group and F/R its free presentation, then

$$\frac{F \cap (1 + \mathfrak{r}^2\mathfrak{f} + \mathfrak{f}^4)}{\gamma_3(R)\gamma_4(F)} \cong L_2\mathfrak{L}^3_s(G_{ab}).$$

In Section 3 we study limits of functors from the category of free presentations of groups to the category of abelian groups, and show their connection with quadratic and cubic functors. Our main results on limits are Theorems 16 and 17 where we show, in particular, that $\varprojlim \frac{\gamma_2(F)}{\gamma_2(R)\gamma_3(F)}$, $\varprojlim \frac{\gamma_3(F)}{\gamma_3(R)\gamma_4(F)}$ and $\varprojlim \frac{\gamma_3(F)}{[\gamma_2(R),\, F]\gamma_4(F)}$ agree with certain derived functors. In particular, we show (see Theorem 17) that

$$\varprojlim \frac{\gamma_3(F)}{[R,\, R,\, F]\gamma_4(F)} \cong L_1\,\mathrm{SP}^3(G_{ab}).$$

For background results, we refer the reader to the monographs [7] and [13] and the article [9].

1. Generalized dimension subgroups

Let F be a free group with basis X. Then the augmentation ideal $\mathfrak{f}$ of the group ring $\mathbb{Z}[F]$ is a two-sided ideal which is free as a left (resp. right) $\mathbb{Z}[F]$-module with basis $\{x - 1 \,|\, x \in X\}$ ([6], p. 32). Thus every element $u \in \mathfrak{f}$ can be written uniquely as

$$u = \sum_{{}_xu\in\mathbb{Z}[F],\, x\in X} {}_xu(x-1) = \sum_{x\in X,\, u_x\in\mathbb{Z}[F]} (x-1)u_x.$$

We refer to ${}_xu$ (resp. u_x) as the *left* (resp. *right*) *partial derivative* of u with respect to the generator x.