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A short note on the multiplier ideals of monomial space curves

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ABSTRACT

Thompson (2014) exhibits a formula for the multiplier ideal with multiplier λ of a monomial curve C with ideal I as an intersection of a term coming from the I-adic valuation, the multiplier ideal of the term ideal of I, and terms coming from certain specified auxiliary valuations. This short note shows it suffices to consider at most two auxiliary valuations. This improvement is achieved through a more intrinsic approach, reduction to the toric case.

1. Introduction

Let (Y, Δ) be a pair, consisting of a normal variety Y over an algebraically closed field of characteristic zero and a $\mathbb{Q}$-divisor Δ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : X \to Y$ be a log resolution of the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ that is also a log resolution of the pair (Y, Δ). That is, π is a proper birational morphism such that X is smooth, the union of the exceptional set of π and $\pi^{-1}(\Delta)$ is a divisor with simple normal crossing support, and $\mathcal{I} \cdot \mathcal{O}_X = \mathcal{O}_X(-F)$ is also a divisor with simple normal crossing support. In this setting, we define the multiplier ideal of $\mathcal{I}^\lambda$ on the pair (Y, Δ) to be

$$\mathfrak{J}\left((Y, \Delta), \mathcal{I}^\lambda\right) = \pi_* \mathcal{O}_X(K_X - \lfloor \pi^*(K_Y + \Delta) + \lambda F \rfloor).$$

This ideal sheaf on Y does not depend upon the choice of log resolution.

In recent years, researchers have begun to study which divisors on a log resolution contribute jumping numbers. See Alberich-Carramiñana, Àlvarez Montaner and Dachs-Cadefau [1], Galindo and Monserrat [6], Hyry and Järvilehto [10], Naie [11,12], Smith and Thompson [14], and Tucker [18]. This paper refines the

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result of Thompson [17] by finding a smaller set of divisors that contains all the divisors that contribute jumping numbers for a monomial space curve.

Section 2 of this paper recalls a strengthening of the notion of an embedded resolution of singularities known as a factorizing resolution and uses it to provide a proposition (Proposition 3 below) about the structure of multiplier ideals.

Section 3 below recalls the Howald–Blickle Theorem (Proposition 4 on page 2461) that provides a formula for the multiplier ideals of a monomial ideal on a normal affine toric variety, provides a reinterpretation (Proposition 6 on page 2461) of that theorem, and provides a formula (Proposition 7 on page 2462) for the multiplier ideals of a principal binomial ideal.

Section 4 on page 2462 applies the ideas of the previous sections to refine the result of Thompson [17].

2. Using factorizing resolutions to compute multiplier ideals

Definition 1. Let Z be a generically smooth subscheme of any variety Y. A *factorizing resolution* of Z is an embedded resolution $\pi : X \to Y$ of Z such that

$$\mathcal{I}_Z \cdot \mathcal{O}_X = \mathcal{I}_{\widetilde{Z}} \cdot \mathcal{L}$$

where $\widetilde{Z}$ is the strict transform of Z, $\mathcal{L}$ is an invertible sheaf, and the support of $\mathcal{I}_Z \cdot \mathcal{O}_X$ is a simple normal crossings variety.

Recall that π is an embedded resolution of Z if it is proper birational morphism $\pi : X \to Y$ such that: X is smooth and π is an isomorphism over the generic points of the components of Z, the exceptional locus $\operatorname{exc}(\pi)$ of π is a divisor with simple normal crossing support, and the strict transform $\widetilde{Z}$ is smooth and transverse to $\operatorname{exc}(\pi)$. For an embedded resolution, we always have $\mathcal{I}_Z \cdot \mathcal{O}_X = \mathcal{I}_{\widetilde{Z}} \cap \mathcal{L}$ for some invertible sheaf $\mathcal{L}$. Here we require the intersection to be a product. Typically, this is achieved by blowing up embedded components of $\mathcal{I}_Z \cdot \mathcal{O}_X$. Here is a theorem on the existence of factorizing resolutions.

Proposition 2. *(See Theorem 1.2 of Bravo [3], Section 3 of Eisenstein [5].) Let Z be a generically smooth subscheme of any variety Y over an algebraically closed field of characteristic zero such that there exists a birational morphism $\mu_1 : Y' \to Y$ from a smooth variety Y' that is an isomorphism over the generic points of the components of Z. If D is a divisor on Y' with simple normal crossing support such that no component of the strict transform of Z is contained in D, then there exists a factorizing resolution $\pi : X \to Y$ of Z that factors through μ_1, $\pi = \mu_2 \circ \mu_1$, such that $\widetilde{Z} \cup \operatorname{exc}(\pi) \cup \mu_2^{-1}(D)$ has simple normal crossing support.*

Notice that if $\pi : X \to Y$ is such a factoring resolution of Z, then the blowup of $\widetilde{Z}$ is a log resolution of Z and that the exceptional locus of this blowup consists of a collection of prime divisors in one-to-one correspondence with the components of Z with codimension at least two.

Proposition 3. *Let $Z_1, \ldots, Z_r$ be the components of Z and suppose e_i is the codimension of Z_i for all i. Fix a factorizing resolution $\pi : X \to Y$ of Z that is also a log resolution of the pair (Y, Δ) and let $\mathfrak{b} = \pi_*(\mathcal{L})$ where $\mathcal{I}_Z \cdot \mathcal{O}_X = \mathcal{I}_{\widetilde{Z}} \cdot \mathcal{L}$ as above. Then,*

$$\mathfrak{J}\left((Y,\Delta), \mathcal{I}_Z^{\lambda}\right) = \mathfrak{J}\left((Y,\Delta), \mathfrak{b}^{\lambda}\right) \cap \bigcap_{i=1}^{r} \mathcal{I}_{Z_i}^{(\lfloor \lambda+1-e_i \rfloor)}$$

Proof. Since $\mathcal{I}_Z \subseteq \mathfrak{b}$, it is clear that $\mathfrak{J}\left((Y,\Delta), \mathcal{I}_Z^{\lambda}\right) \subseteq \mathfrak{J}\left((Y,\Delta), \mathfrak{b}^{\lambda}\right)$. Let us now show $\mathfrak{J}\left((Y,\Delta), \mathcal{I}_Z^{\lambda}\right) \subseteq \mathcal{I}_{Z_i}^{(\lfloor \lambda+1-e_i \rfloor)}$ for each i. Since Z is generically smooth, the $\mathcal{I}_{Z_i}$ are prime and the $\mathcal{I}_{Z_i}^{(\lfloor \lambda+1-e_i \rfloor)}$ are primary.