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Symmetric semi-algebraic sets and non-negativity of symmetric polynomials

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ABSTRACT

The question of how to certify the non-negativity of a polynomial function lies at the heart of Real Algebra and has important applications to optimization. Building on work by Choi, Lam, and Reznick [4], as well as Harris [5], Timofte [9] provided a remarkable method to efficiently certify non-negativity of symmetric polynomials. In this note we slightly generalize Timofte's statements and investigate families of polynomials that allow special representations in terms of power-sum polynomials. We also recover the consequences of Timofte's original statements as a corollary.

1. Introduction

Real Algebraic Geometry evolved around the question how to certify that a polynomial function $f \in \mathbb{R}[X_1, \dots, X_n]$ assumes only nonnegative values and the study of so-called semi-algebraic sets. These subsets of $\mathbb{R}^n$ are defined by a Boolean formula whose atoms are polynomial equalities and inequalities. Given polynomials $f_1 \dots, f_m \in \mathbb{R}[X_1, \dots, X_n]$ we will denote by $S(f_1, \dots, f_m) \subset \mathbb{R}^n$ a semi-algebraic set, such that the equalities and inequalities appearing in the description are given by the polynomials $f_1, \dots, f_m$. With this setup, problems such as deciding if a semi-algebraic set is empty or not, or computing topological invariants of such sets are central to Real Algebraic Geometry [1]. These algorithmic questions also have applications to other areas of mathematics, for example to optimization. There has been some interest in this question in particularly structured situations, for example polynomials invariant under the action of a group. In this note we investigate symmetric polynomials, i.e., polynomials invariant under all permutations of the variables. In this setting the following so-called *half-degree principle* (see [7,9]) applies: Let f be a symmetric polynomial in n real variables of degree $2d$. Then, if the inequality $f(y) \geq 0$ holds on all points $y \in \mathbb{R}^n$ that do not have more than d distinct coordinates, it is valid for all $y \in \mathbb{R}^n$. Furthermore, the so-called *degree principle* applies to a general semi-algebraic set S described by symmetric polynomials $f_1, \dots, f_m$ such that every symmetric polynomial f_i has at most degree d. In this case, emptiness of S can be certified by

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restricting to the points with at most d distinct coordinates. Both the half-degree principle and the degree principle allow for a more efficient way to check for non-negativity of a symmetric polynomial or to certify emptiness of a symmetric semi-algebraic set. For $k \in \mathbb{Z}$ a k-partition of n (denoted as $\vartheta \vdash_k n$) is an ordered sequence $\vartheta := (\vartheta_1, \ldots, \vartheta_k)$ of non-negative integers with $\vartheta_1 \geq \vartheta_2 \geq \ldots, \geq \vartheta_k$ and $\vartheta_1 + \vartheta_2 + \ldots + \vartheta_k = n$. Given a symmetric polynomial f and $\vartheta \vdash_k n$ one can define a k-variate polynomial $f^\vartheta \in \mathbb{R}[T_1, \ldots, T_k]$ via

$$f^\vartheta(T_1, \ldots, T_k) := f(\underbrace{T_1, \ldots, T_1}_{\vartheta_1}, \underbrace{T_2, \ldots, T_2}_{\vartheta_2}, \ldots, \underbrace{T_k, \ldots, T_k}_{\vartheta_k}).$$

Using this notation the half-degree principle states that if f^ϑ is non-negative on $\mathbb{R}^k$ for all $\vartheta \vdash_k n$, then the original symmetric polynomial f is non-negative. Since $\vartheta \vdash_k n$ implies that $\vartheta \in \{1, \ldots, n\}^k$, the number of possible k partitions of n is bounded by n^k. Hence the complexity of deciding if a symmetric polynomial of a fixed degree is non-negative depends only polynomially on n. This construction can be applied appropriately to any semi-algebraic set S defined by symmetric polynomials whose degrees are bounded by a fixed number. This idea has remarkable consequences, for example in SDP-relaxations for optimization tasks defined by symmetric polynomials [3,8], or the study of topological complexity of projections of general (i.e. non-symmetric) semi-algebraic sets [2]. In the remainder of this paper we want to show that these remarkable statements and a slight generalization can be derived in an elementary way by properties of the power sum polynomials.

2. Generalizing the degree principle

One of the main observations in the proof of the degree principle is the form of the representation of a symmetric polynomial of degree d in terms of generators of the polynomial algebra of symmetric polynomials: For integers $n, i \in \mathbb{N}$ define the power sum polynomials

$$p_i^{(n)} := \sum_{j=1}^{n} X_j^i.$$

We will omit the superscript n whenever the number of variables is clear. The following statement is well-known (see for example [6]) and sometimes referred to as the fundamental theorem of symmetric polynomials.

Theorem 2.1. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be symmetric, then there is a unique polynomial $g \in \mathbb{R}[Z_1, \ldots, Z_n]$ such that*

$$f(X_1, \ldots, X_n) = g(p_1(X_1, \ldots, X_n), \ldots, p_n(X_1, \ldots, X_n)). \tag{2.1}$$

Since the decomposition in (2.1) is unique, a closer inspection gives the following.

Corollary 2.2. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be symmetric of degree $d \leq n$. Then setting $k = \lfloor \frac{d}{2} \rfloor$ and $d' = \min\{d, n\}$, we have*

$$f(X_1, \ldots, X_n) = g_0(p_1(X_1, \ldots, X_n), \ldots, p_k(X_1, \ldots, X_n))$$
$$+ \sum_{j=k+1}^{d'} g_{j-k}(p_1(X_1, \ldots, X_n), \ldots, p_k(X_1, \ldots, X_n)) \cdot p_j(X_1, \ldots, X_n)$$

The following definition generalizes this property to symmetric polynomials that are representable using few power sums.