Journal of Pure and Applied Algebra 220 (2016) 2846–2851

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

The piecewise Noetherian property in power series rings over a valuation domain ☆

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ARTICLE INFO

Article history:
Received 20 December 2015
Received in revised form 25 January 2016
Available online 3 February 2016
Communicated by A.V. Geramita

MSC:
13E99; 13F25; 13F30

ABSTRACT

Let V be a valuation domain and let P be a nonzero prime ideal of V. We characterize when $V[\![X]\!]_{P[\![X]\!]}$ is a valuation domain and when it is a Noetherian ring. We then show that $V[\![X]\!]$ is piecewise Noetherian if and only if V is Noetherian. As a corollary, we obtain that the piecewise Noetherian property is not preserved under the power series extension.

1. Introduction

Let R be a commutative ring with unity. An ideal I of R is said to have finite-ideal length if the ring R_S/IR_S is an Artinian ring, where $S = R \setminus \bigcup\{P \mid P$ is a prime ideal minimal over $I\}$. The ring R is called piecewise Noetherian if all its ideals have finite-ideal lengths and the set of prime ideals of R satisfies the ascending chain condition, or equivalently, if R has Noetherian spectrum and it has a.c.c. on P-primary ideals for each prime ideal P.

One interesting question is whether the integral closure of a Noetherian domain is piecewise Noetherian. According to [4, Proposition 3.7], it is equivalent to ask whether the maximal ideals of the integral closure of a Noetherian domain are finitely generated. This question was posed by W. Heinzer in 1973 [8, Question] and still remains open.

In [4], J.A. Beachy and W.D. Weakley showed that if R is piecewise Noetherian, then so is the polynomial ring $R[X]$. It is natural to ask whether the piecewise Noetherian property is preserved under the power series extension. This question was posed by H. Kim in his talk in Graz conference in 2014. We answer the question in the negative by investigating the power series ring over a valuation domain.

☆ This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (No. 2015R1C1A2A01055124).

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http://dx.doi.org/10.1016/j.jpaa.2016.01.004

For a nonzero prime ideal P of a domain D, J.T. Arnold and J.W. Brewer considered in [3] the problem of when $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain. Their main result states that if $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain, then D_P is a 1-dimensional discrete valuation ring (DVR). Moreover, in this case, $D[\![X]\!]_{P[\![X]\!]}$ is also a 1-dimensional DVR. They showed however that the converse is false. They also found a sufficient condition, but were unable to find a nice condition which is both necessary and sufficient.

We start by giving a characterization of when $V[\![X]\!]_{P[\![X]\!]}$ is a valuation domain for a valuation domain V and a nonzero prime ideal P. Using the characterization, we show that $V[\![X]\!]_{P[\![X]\!]}$ is a valuation domain if and only if it is a Noetherian ring. Next, using this equivalence, we show that $V[\![X]\!]$ is a piecewise Noetherian ring if and only if V is Noetherian. Since there exists a piecewise Noetherian valuation domain V which is not Noetherian, this gives the negative answer to the above mentioned question.

2. Main results

According to Arnold and Brewer [3], if $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain, then D_P is a 1-dimensional DVR. Conversely, if D_P is a 1-dimensional DVR and $P[\![X]\!] = PD[\![X]\!]$, then $D[\![X]\!]_{P[\![X]\!]}$ is a valuation domain. They give an example demonstrating that the latter condition is not necessary in general, but we will see below that this condition is in fact necessary when $D = V$ is a valuation domain. Moreover, assuming that V_P is a 1-dimensional DVR, it will become clear that the condition $P[\![X]\!] = PV[\![X]\!]$ is satisfied precisely if either P is maximal or P is the intersection of the prime ideals which properly contain it, but not a countable intersection of prime ideals.

We start with a lemma, which follows easily from [7, Theorem 1] by applying the theorem to V/P, but we include a brief proof for the sake of completeness.

Lemma 1. *Let V be a valuation domain and let P be a prime ideal of V. Then for each sequence $\{s_i\}_{i=0}^{\infty}$ of elements of $V \setminus P$, $\bigcap_{i=0}^{\infty}(s_i) \neq P$ if and only if $V_P[\![X]\!] = V[\![X]\!]_{V\setminus P}$.*

Proof. Assume that for each sequence $\{s_i\}_{i=0}^{\infty}$ of elements of $V \setminus P$, $\bigcap_{i=0}^{\infty}(s_i) \neq P$. Let $f \in V_P[\![X]\!]$. Write $f = \frac{a_0}{s_0} + \frac{a_1}{s_1}X + \frac{a_2}{s_2}X^2 + \cdots$, where $a_i \in V$ and $s_i \in V \setminus P$, $i \geq 0$. By assumption, $\bigcap_{i=0}^{\infty}(s_i) \neq P$. Choose an element $s \in \bigcap_{i=0}^{\infty}(s_i) \setminus P$. Then since $\frac{s}{s_i} \in V$ for each $i \geq 0$, we have $sf \in V[\![X]\!]$ and hence $f \in V[\![X]\!]_{V\setminus P}$.

Conversely, assume that $V_P[\![X]\!] = V[\![X]\!]_{V\setminus P}$. Let $\{s_i\}_{i=0}^{\infty}$ be a sequence of elements of $V \setminus P$. Since $\frac{1}{s_0} + \frac{1}{s_1}X + \frac{1}{s_2}X^2 + \cdots \in V_P[\![X]\!] = V[\![X]\!]_{V\setminus P}$, there exists an element $s \in V \setminus P$ such that $s(\frac{1}{s_0} + \frac{1}{s_1}X + \frac{1}{s_2}X^2 + \cdots) \in V[\![X]\!]$. Then $P \subsetneq (s) \subseteq \bigcap_{i=0}^{\infty}(s_i)$. □

Remark 2. Let V be a valuation domain and let P be a nonmaximal prime ideal. If V satisfies the above equivalent conditions, then V has no prime ideal just above P, i.e., no prime ideal Q such that $Q \supset P$ and $\text{ht}(Q/P) = 1$. However, the converse does not hold.

Example 3. (1) There exists a valuation domain V (which is not a field) such that $V_{V\setminus(o)}[\![X]\!] = V[\![X]\!]_{V\setminus(0)}$. See [7, Example].

(2) Let H be a value group of the valuation domain V in Gilmer's example [7, Example] and let G be the direct sum of the groups $\mathbb{Z}$ and H. Give the group $G = \mathbb{Z} \oplus H$ the lexicographic order. Then G is a totally ordered group. Let W be a valuation domain with value group G and let P be the prime ideal corresponding to the convex subgroup $\{0\} \oplus H$ of G. Then $\text{ht}\, P = 1$, $P \neq P^2$, and for each sequence $\{s_i\}_{i=0}^{\infty}$ of elements of $W \setminus P$, $\bigcap_{i=0}^{\infty}(s_i) \neq P$.

(3) Let A be the set of nonnegative real numbers (or the set of nonnegative rational numbers), let each Z_α, $\alpha \in A$, be the integer group, and let $G = \bigoplus_{\alpha\in A} \mathbb{Z}_\alpha$ be the weak lexicographic sum of the groups $\mathbb{Z}_\alpha$. Let V be a valuation domain with value group G and let P be the prime ideal of V corresponding to the convex subgroup $\{0\} \oplus (\bigoplus_{\alpha>0} \mathbb{Z}_\alpha)$ of G. Then $\text{ht}\, P = 1$, $P \neq P^2$, and V has no prime ideal just above P. However,