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Test sets for nonnegativity of polynomials invariant under a finite reflection group

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ABSTRACT

A set $S \subset \mathbb{R}^n$ is a nonnegativity witness for a set U of homogeneous polynomials if F in U is nonnegative on $\mathbb{R}^n$ if and only if it is nonnegative at all points of S. We prove that the union of the hyperplanes perpendicular to the elements of a root system $\Phi \subseteq \mathbb{R}^n$ is a witness set for nonnegativity of forms of low degree which are invariant under the reflection group defined by Φ. We prove that our bound for the degree is sharp for all reflection groups which contain multiplication by -1. We then characterize subspaces of forms of arbitrarily high degree where this union of hyperplanes is a nonnegativity witness set. Finally we propose a conjectural generalization of Timofte's half-degree principle to finite reflection groups.

1. Introduction

The general problem of deciding whether a homogeneous polynomial F in n variables with real coefficients is nonnegative has been one of the guiding questions of real algebraic geometry since the time of Hilbert. The intrinsic interest of this question has been complemented by a recent surge in its applications to polynomial optimization [2], control theory [11] and to the general theory of approximation algorithms [1] among others. For all these applications, finding simple and efficient certificates for guaranteeing the nonnegativity of a given polynomial is of paramount importance.

In the case of polynomials F which are invariant under symmetric groups, such certificates can take the form of "nonnegativity test sets", that is subsets $S \subset \mathbb{R}^n$ such that nonnegativity of F on S is equivalent to the nonnegativity of F in $\mathbb{R}^n$. If the test set S is sufficiently simple then verifying that F is nonnegative becomes a simpler problem. The work of several authors has provided important and interesting examples of nonnegativity test sets for symmetric polynomials [3,6,16,13,8]. The following theorems are some representative results,

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Theorem *(Choi–Lam–Reznick). (See [3].) An n-ary even symmetric sextic is nonnegative if and only if it is so at the n points* $(1,0,\dots,0)$, $(1,1,0,\dots,0)$,..., $(1,1\dots,1)$.

Theorem *(Timofte's half-degree principle). (See [16,13].) An n-variate symmetric polynomial of degree* $2d$ *is nonnegative if and only if it is so at every point with at most* $\max\{2,d\}$ *distinct components.*

The purpose of this note is to provide test sets for polynomials which are invariant under the action of a finite reflection group $\mathcal{W}$. Our main finding is that the locus of points where a $\mathcal{W}$-invariant form achieves its minimum must contain the real points of certain families of $\mathcal{W}$-invariant curves. These curves, constructed using the classical theory of finite reflection groups, must in turn intersect certain simple subsets which, as a result, can be used as test sets for nonnegativity.

To give a more precise description of the results of the article we need to establish some notation (see Section 2 for additional background and definitions). Let V be a real Euclidean vector space of dimension n and let $\Phi \subseteq V$ be a root system which spans V. Let $\mathcal{W}$ be the finite reflection group defined by Φ. By a theorem of Chevalley (see Theorem 2.1) the algebra of polynomial functions on V which are invariant under $\mathcal{W}$ (i.e. polynomials f such that $f \circ g^{-1} = f$ for all $g \in \mathcal{W}$) is generated by n basic homogeneous invariants of degrees $1 \le d_1 \le \dots \le d_n$. Our first result is the following existence theorem for codimension one test sets,

Theorem 1.1. *Assume* $n > 1$ *and that* Φ *spans* V*. If* F *is a* $\mathcal{W}$*-invariant form of degree* $2d < 2d_n$ *then* F *is nonnegative on* V *if and only if* F *is nonnegative on the set* $\mathcal{H}_{\mathcal{W}}$*, defined as the union of the hyperplanes that are perpendicular to the elements of* Φ.

If $\Phi := \{e_i - e_j : 1 \le i \ne j \le n\} \subseteq \mathbb{R}^n$ then the group $\mathcal{W}$ is the permutation group in n letters acting by permuting the components. Its basic invariants have degrees $1, 2, \dots, n$ and Theorem 1.1 implies that a symmetric form F of degree $2d < 2n$ is nonnegative if and only if it is so on the hyperplanes $x_i - x_j = 0$ (i.e. if and only if F is nonnegative at all points with at most $n-1$ distinct components). If $\Phi = \{\pm e_i : 1 \le i \le n\} \cup \{\pm e_i \pm e_j : 1 \le i \ne j \le n\} \subseteq \mathbb{R}^n$ then $\mathcal{W}$ is the Weyl group of B_n acting on the polynomial ring in n variables by signed permutations. The degrees of the basic invariants of $\mathcal{W}$ are $2, 4, \dots, 2n$. In this case Theorem 1.1 says that an even symmetric form of degree $2d < 4n$ is nonnegative if and only if it is so on the hyperplanes defined by $x_i = 0$ and by $\pm x_i \pm x_j = 0$ for $1 \le i \ne j \le n$. The statement of Theorem 1.1 can be made more explicit by knowing the degrees of the basic invariants. Table 1 contains these degrees for the reflection groups $\mathcal{W}$ generated by an irreducible root system (see [7, Chapter 2] for a proof of the classification of finite reflection groups and for explicit descriptions of the roots in each case).

Next, it is natural to ask whether the degree bound in Theorem 1.1 is sharp. We prove that this is the case when $\mathcal{W}$ is a reflection group all of whose basic invariants have even degree (i.e. for those reflection groups containing multiplication by -1). More generally, we prove the following existence theorem for $\mathcal{W}$-invariant forms for which the set $\mathcal{H}_{\mathcal{W}}$ is *not* a test set for nonnegativity.

Theorem 1.2. *Let* $\underline{o}$ *and* $\overline{o}$ *be the smallest and largest odd degrees of basic invariants for* $\mathcal{W}$ *and define* $\underline{o} = \overline{o} = 1$ *if all invariants of* $\mathcal{W}$ *have even degree. If* $2d \ge \max(2d_n, 2(\underline{o}+\overline{o}))$ *then there exists a* $\mathcal{W}$*-invariant form* F *of degree* $2d$ *which is nonnegative on* $\mathcal{H}_{\mathcal{W}}$ *and which is negative at some point of* V.

In the case of irreducible reflection groups the bounds from the previous two theorems can be computed explicitly and we do so in Table 1. Theorem 1.2 implies that Theorem 1.1 is sharp whenever the entries of the last two columns of Table 1 agree.

By Theorem 1.2 we know that the set $\mathcal{H}_{\mathcal{W}}$ above is not a test set for *all* forms of sufficiently large degree. It is therefore natural to ask whether some subspaces of forms of higher degree have other nonnegativity