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Annihilators in zero-divisor graphs of semilattices and reduced commutative semigroups

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ABSTRACT

Let $V(G)$ be the set of vertices of a simple connected graph G. The set $\mathcal{L}_1(G)$ consisting of $\emptyset$, $V(G)$, and all neighborhoods $N(v)$ of vertices $v \in V(G)$ is a subposet of the complete lattice $\mathcal{L}(G)$ (under inclusion) of all intersections of elements in $\mathcal{L}_1(G)$. In this paper, it is shown that $\mathcal{L}_1(G)$ is a join-semilattice and $\mathcal{L}(G)$ is a Boolean algebra if and only if G is realizable as the zero-divisor graph of a meet-semilattice with 0. Also, if $\mathcal{L}_1(G)$ is a meet-semilattice and $\mathcal{L}(G)$ is a Boolean algebra, then G is realizable as the zero-divisor graph of a join-semilattice with 0. As a corollary, graphs that are realizable as zero-divisor graphs of commutative semigroups with 0 that do not have any nonzero nilpotent elements are classified.

1. Introduction

Given a (multiplicative) commutative semigroup S with 0, the *zero-divisor graph* $\Gamma(S)$ of S is the (undirected) graph whose vertices are the nonzero zero-divisors of S such that two distinct vertices x and y are adjacent (via a single edge) if and only if $xy = 0$. The concept of a zero-divisor graph was introduced by I. Beck in [6]. In Beck's zero-divisor graph, the vertices were the elements of a commutative ring, and his study was focused on colorings. Of course, the zero-divisor relations involving 0 or any nonzero-divisor are fully understood. Therefore, when zero-divisor graphs are studied in order to illuminate algebraic structure, it is customary to restrict the set of vertices to only include the nonzero zero-divisors of S. This approach was first taken by D.F. Anderson and P.S. Livingston in [5] while studying commutative rings, and was extended to commutative semigroups with 0 by F.R. DeMeyer, T. McKenzie, and K. Schneider in [11]. Surveys on zero-divisor graphs are provided in [2] and [8].

Recently, the zero-divisor graph concept has been extended to *posets* (that is, to partially ordered sets; see [12,14,17,19]). Let $\mathcal{P}$ be a poset with a least element 0. Given a subset $\emptyset \neq A \subseteq \mathcal{P}$, define $A^\wedge = \{x \in \mathcal{P} \mid x \leq a$ for every $a \in A\}$. The *zero-divisor graph* of $\mathcal{P}$, denoted by $\Gamma(\mathcal{P})$, is the (undirected) graph whose vertices are the nonzero elements of $Z(\mathcal{P}) = \big\{x \in \mathcal{P} \mid \{x, y\}^\wedge = \{0\}$ for some $0 \neq y \in \mathcal{P}\big\}$ such that two vertices

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x and y are adjacent (via a single edge) if and only if $\{x, y\}^{\wedge} = \{0\}$ (equivalently, the infimum $x \wedge y$ of $\{x, y\}$ exists in $\mathcal{P}$ and is 0).

Although zero-divisor graphs of certain posets have been considered earlier (e.g., zero-divisor graphs of Boolean algebras were considered in [4]), zero-divisor graphs of general posets were first studied by R. Halaš and M. Jukl in [14]. As with Beck's paper, they allowed every element of $\mathcal{P}$ to be a vertex, and they were mainly interested in colorings. The present definition was used by D. Lu and T. Wu in [19]. Zero-divisor graphs of posets were extended to *quasiordered* sets (i.e., sets endowed with a relation that is reflexive and transitive) by R. Halaš and H. Länger in [15], and the case when $\mathcal{P}$ is a lattice was later studied by E. Estaji and K. Khashyarmanesh in [12].

The present paper focuses on semilattices with a least element 0. Sections 2 and 3 extend the Introduction by providing brief expositions on the essential topics that will be further developed in this article. Sections 4, 5, and 6 continue the investigations from [17] and [19].

By imposing additional conditions to a characterization of zero-divisor graphs of posets given in [19], the zero-divisor graphs of meet-semilattices are completely determined, and sufficient conditions are provided to determine whether a graph is realizable as the zero-divisor graph of a join-semilattice (Theorems 4.4 and 5.1). As a corollary, zero-divisor graphs of commutative semigroups without nonzero nilpotent elements are classified (Corollary 1.2). Along the way, a graph-theoretic analogue of the usual ring-theoretic *annihilator* is developed, and is used to organize graph-theoretic criteria in an algebraic form (e.g., see Section 3).

The set of vertices of a graph G will be denoted by $V(G)$. Two simple graphs G_1 and G_2 are called *isomorphic*, written $G_1 \cong G_2$, if there exists a bijection $\varphi : V(G_1) \to V(G_2)$ such that two vertices x and y are adjacent in G_1 if and only if $\varphi(x)$ and $\varphi(y)$ are adjacent in G_2.

Let G be a simple graph. If $v \in V(G)$, then let $\mathbf{c}_G(v) = \{w \in V(G) \mid w \text{ is adjacent to } v\}$. Given any $A \subseteq V(G)$, define $\mathbf{c}_G(A) = V(G)$ if $A = \emptyset$, and otherwise let $\mathbf{c}_G(A) = \cap\{\mathbf{c}_G(v) \mid v \in A\}$. When there is no risk of confusion, the set $\mathbf{c}_G(A)$ will be denoted by $\mathbf{c}(A)$. (The choice in notation follows from the fact that "$\mathbf{c}$" is a complementation on the lattice $\mathcal{L}(G) = \{\mathbf{c}(A) \mid A \subseteq V(G)\}$ under inclusion; see Section 3.)

The *neighborhood* in G of a set $\emptyset \neq A \subseteq V(G)$ is the set $N_G(A) = \cup\{\mathbf{c}_G(v) \mid v \in A\}$. When $A = \{a_1, \ldots, a_n\}$ is a finite set, then we will write $N_G(A) = N_G(a_1, \ldots, a_n)$. Of course, $N_G(a) = \mathbf{c}_G(a)$ for any vertex a of G. For consistency, we will always write $\mathbf{c}_G(a)$ instead of $N_G(a)$. Also, when there is no risk of confusion, the set $N_G(A)$ will be denoted by $N(A)$.

Let $\mathcal{L}_1(G) = \{\mathbf{c}(a) \mid a \in V(G)\} \cup \{\emptyset, V(G)\} \subseteq \mathcal{L}(G)$. The following corollary is proved in Section 5.

Corollary 1.1. *Let G be a simple connected graph with $2 \leq |V(G)| < \infty$. Then the following statements are equivalent.*

(1) *$\mathcal{L}_1(G)$ is a lattice, and $\mathcal{L}(G)$ is a Boolean algebra.*
(2) *$\mathcal{L}_1(G)$ is a meet-semilattice, and $\mathcal{L}(G)$ is a Boolean algebra.*
(3) *$\mathcal{L}_1(G)$ is a join-semilattice, and $\mathcal{L}(G)$ is a Boolean algebra.*
(4) *$G \cong \Gamma(\mathcal{S})$ for some lattice $\mathcal{S}$.*
(5) *$G \cong \Gamma(\mathcal{S})$ for some meet-semilattice $\mathcal{S}$.*
(6) *$G \cong \Gamma(\mathcal{S})$ for some join-semilattice $\mathcal{S}$.*

The next corollary summarizes several of the main results of this paper, and is proved in Section 6.

Corollary 1.2. *Let G be a simple connected graph with $|V(G)| \geq 2$. Then the following statements are equivalent.*

(1) *$G \cong \Gamma(S)$ for some reduced commutative semigroup S.*
(2) *$G \cong \Gamma(S)$ for some commutative Boolean semigroup S.*