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On conciseness of words in profinite groups

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ABSTRACT

Let w be a group word. It is conjectured that if w has only countably many values in a profinite group G, then the verbal subgroup $w(G)$ is finite. In the present paper we confirm the conjecture in the cases where w is a multilinear commutator word, or the word x^2, or the word $[x^2, y]$.

1. Introduction

Let $w = w(x_1, \dots, x_k)$ be a group-word, and let G be a group. The verbal subgroup $w(G)$ of G determined by w is the subgroup generated by the set of all values $w(g_1, \dots, g_k)$, where $g_1, \dots, g_k$ are elements of G. A word w is said to be concise if whenever the set of its values is finite in G, it always follows that the subgroup $w(G)$ is finite. More generally, a word w is said to be concise in a class of groups X if whenever the set of its values is finite in a group $G \in X$, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise, but later Ivanov proved that this problem has a negative solution in its general form [5] (see also [10, p. 439]). On the other hand, many relevant words are known to be concise.

For instance, it is an easy observation by P. Hall that every non-commutator word is concise (see e.g. [12, Lemma 4.27]). A word w is non-commutator if the sum of exponents of at least one variable involved in w is non-zero. It was shown in [15] that the multilinear commutator words are concise (see also [3]). Such words are also known under the name of outer commutator words and are precisely the words that can be written in the form of multilinear Lie monomials. Merzlyakov showed that every word is concise in the class of linear groups [8] while Turner-Smith proved that every word is concise in the class of residually finite groups all of whose quotients are again residually finite [14]. It was shown in [1] that if w is a multilinear commutator word and n is a prime-power, then the word w^n is concise in the class of residually finite groups. Another interesting family of words that are concise in residually finite groups was exhibited in [4].

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There is an open problem, due to Jaikin-Zapirain [6], whether every word is concise in the class of profinite groups. Of course, the verbal subgroup $w(G)$ in a profinite group G is defined as the closed subgroup generated by all w-values. In the present paper we deal with a newly discovered phenomenon that suggests that the very definition of conciseness in profinite groups can perhaps be relaxed. More precisely, we provide evidence in support of the following conjecture.

Conjecture. *Assume that the word w has only countable many values in a profinite group G. Then the verbal subgroup $w(G)$ is finite.*

We show that the above conjecture holds true whenever w is multilinear commutator word. Thus, our first result is as follows.

Theorem 1.1. *Let w be a multilinear commutator word and G a profinite group having only countably many w-values. Then the verbal subgroup $w(G)$ is finite.*

Further, we attempt to deal with the conjecture in the case where w is a non-commutator word. Recall that the "classical" conciseness of such words is just an easy observation. In sharp contrast, the above conjecture for non-commutator words seems hard to deal with. So far we succeeded only in the case of the word $w = x^2$.

Theorem 1.2. *Let G be a profinite group having only countably many squares. Then G^2 is finite.*

Here, as usual, G^n denotes the closed subgroup of G generated by n-th powers.

We were also able to confirm the conjecture for the commutator word $w = [x^2, y]$.

Theorem 1.3. *Let G be a profinite group having only countably many values of the word $w = [x^2, y]$. Then the verbal subgroup $w(G)$ is finite.*

Throughout the paper by a subgroup of a profinite group we mean a closed subgroup and we say that a subgroup is generated by a set S to mean that it is topologically generated by S.

2. Multilinear commutator words

A multilinear commutator word (outer commutator word) is a word which is obtained by nesting commutators, but using always different variables. Thus the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator while the Engel word $[x_1, x_2, x_2, x_2]$ is not. An important family of multilinear commutator words is formed by the derived words δ_k, on 2^k variables, which are defined recursively by

$$\delta_0 = x_1, \qquad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$, the k-th derived subgroup of G. Another distinguished family of multilinear commutators are the simple commutators γ_k, given by

$$\gamma_1 = x_1, \qquad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \ldots, x_k].$$

The corresponding verbal subgroups $\gamma_k(G)$ are the terms of the lower central series of G.

Recall that a group is periodic (torsion) if every element of the group has finite order and a group is called locally finite if each of its finitely generated subgroups is finite. Periodic profinite groups have received a good deal of attention in the past. In particular, using Wilson's reduction theorem [16], Zelmanov has