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Vertices of Lie modules

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ABSTRACT

Let $\mathrm{Lie}_F(n)$ be the Lie module of the symmetric group $\mathfrak{S}_n$ over a field F of characteristic $p > 0$, that is, $\mathrm{Lie}_F(n)$ is the left ideal of $F\mathfrak{S}_n$ generated by the Dynkin–Specht–Wever element ω_n. We study the problem of parametrizing non-projective indecomposable summands of $\mathrm{Lie}_F(n)$, via describing their vertices and sources. Our main result shows that this can be reduced to the case when n is a power of p. When $n = 9$ and $p = 3$, and when $n = 8$ and $p = 2$, we present a precise answer. This suggests a possible parametrization for arbitrary prime powers.

1. Introduction

The Lie module of the symmetric group $\mathfrak{S}_n$ occurs in various contexts within algebra and topology, where the name-giving property is its close relation to the free Lie algebra; for more details, see for example the introduction in [14]. In the present paper, letting F be an algebraically closed field of characteristic $p > 0$, we realize the Lie module $\mathrm{Lie}_F(n)$ of $\mathfrak{S}_n$, for $n \geqslant 2$, as the submodule $F\mathfrak{S}_n\omega_n$ of the regular $F\mathfrak{S}_n$-module, where

$$\omega_n := (1 - c_2)(1 - c_3) \cdots (1 - c_n) \in F\mathfrak{S}_n$$

is the Dynkin–Specht–Wever element of $F\mathfrak{S}_n$, where in turn $c_k \in \mathfrak{S}_n$ is the backward cycle $(k, k-1, \ldots, 2, 1)$. Moreover, $\dim(\mathrm{Lie}_F(n)) = (n-1)!$; see A.5.

1.1. It is well known that $\omega_n^2 = n\omega_n$. Hence if p does not divide n, then $\omega_n/n \in F\mathfrak{S}_n$ is an idempotent, so that $\mathrm{Lie}_F(n)$ is then a direct summand of the regular $F\mathfrak{S}_n$-module and is, thus, projective. In the present paper we are interested in the case when p divides n, which we assume from now on in this section. Then

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$\mathrm{Lie}_F(n)$ cannot be projective; for otherwise $\dim(\mathrm{Lie}_F(n)) = (n-1)!$ would have to be divisible by the p-part of $n!$, which is not the case. Therefore, in this case $\mathrm{Lie}_F(n)$ admits a decomposition

$$\mathrm{Lie}_F(n) = \mathrm{Lie}_F^{\mathrm{pr}}(n) \oplus \mathrm{Lie}_F^{\mathrm{pf}}(n),$$

where $\mathrm{Lie}_F^{\mathrm{pr}}(n)$ is a projective $F\mathfrak{S}_n$-module and where $\mathrm{Lie}_F^{\mathrm{pf}}(n) \neq \{0\}$ is a projective-free $F\mathfrak{S}_n$-module.

The asymptotic behaviour of the quotient $\dim(\mathrm{Lie}_F^{\mathrm{pr}}(n))/\dim(\mathrm{Lie}_F(n))$ has recently been studied by Erdmann and Tan [14], and by Bryant, Lim and Tan [5]. By [5, Thm. 1.2], one has

$$\frac{\dim(\mathrm{Lie}_F^{\mathrm{pr}}(n))}{\dim(\mathrm{Lie}_F(n))} \longrightarrow 1,$$

as $n \longrightarrow \infty$ in $\mathbb{N} \smallsetminus \{p^k \mid k \geqslant 0\}$. Moreover, it is conjectured in [5] that this should remain true when allowing n to vary over all natural numbers. This suggests that $\mathrm{Lie}_F^{\mathrm{pf}}(n)$ should be small, compared with the entire Lie module $\mathrm{Lie}_F(n)$.

Moreover, by work of Erdmann and Tan [15], we also know that the projective-free part $\mathrm{Lie}_F^{\mathrm{pf}}(n)$ of $\mathrm{Lie}_F(n)$ always belongs to the principal block of $F\mathfrak{S}_n$, and Bryant and Erdmann [4] have studied indecomposable direct sum decompositions of the, necessarily projective, part of $\mathrm{Lie}_F(n)$ not contained in the principal block of $F\mathfrak{S}_n$. This leaves open, next to $\mathrm{Lie}_F^{\mathrm{pf}}(n)$, only the direct sum decompositions of the component of $\mathrm{Lie}_F^{\mathrm{pr}}(n)$ belonging to the principal block of $F\mathfrak{S}_n$. We denote the principal block component of $\mathrm{Lie}_F(n)$ by $\mathrm{Lie}_F^{\mathrm{pbl}}(n)$.

1.2. One key ingredient of our approach is a decomposition theorem, expressing $\mathrm{Lie}_F(n)$ as a direct sum of pieces related to Lie modules $\mathrm{Lie}_F(p^d)$, for various d such that p^d divides n. This is obtained by translating the Bryant–Schocker decomposition theorem [6] for Lie powers to Lie modules, using work of Lim and Tan [24]. This paves the way to reduce questions on Lie modules to the case when n is a power of p, and puts the Lie modules $\mathrm{Lie}_F(p^d)$ into the focus of study. In particular, one is tempted to ask whether there is a neat description of the indecomposable direct summands of $\mathrm{Lie}_F(n)$ in terms of those of $\mathrm{Lie}_F(p^d)$, where d varies as indicated above. This has been fully accomplished for the case where p divides n but p^2 does not, with a different line of reasoning, by Erdmann and Schocker [13], while the general case remains a mystery and is subject to further investigations.

Very little information concerning the decomposition of the principal block component of $\mathrm{Lie}_F(p^d)$ is available in the literature, and the projective-free part $\mathrm{Lie}_F^{\mathrm{pf}}(p^d)$ is very poorly understood, even for very small exponents d: to our knowledge, the only cases dealt with systematically are the modules $\mathrm{Lie}_F^{\mathrm{pf}}(p)$, that is, the case $d = 1$, by Erdmann and Schocker [13]; and, apart from the easy case $\mathrm{Lie}_F(4) = \mathrm{Lie}_F^{\mathrm{pf}}(4)$, there are just partial results for $\mathrm{Lie}_F(8)$, by Selick and Wu [33]. The aim of this paper now is to investigate indecomposable direct summands of $\mathrm{Lie}_F(p^d)$, for a few further small values of p and d.

The major obstacle here is that, due to the exponential growth of the dimension of Lie modules in terms of n, these modules quickly become very large. Hence, to proceed further in this direction, we apply computational techniques. More precisely, by this approach we are now able to give a complete description of the Lie modules $\mathrm{Lie}_F(8)$ of dimension 5040, and $\mathrm{Lie}_F(9)$ of dimension 40 320.

Actually, in both cases it turns out that the projective-free part of the Lie module is already indecomposable, where $\mathrm{Lie}_F^{\mathrm{pf}}(8)$ has dimension 816, and $\mathrm{Lie}_F^{\mathrm{pf}}(9)$ has dimension 1683. In view of these results, and those on $\mathrm{Lie}_F^{\mathrm{pf}}(4)$ and $\mathrm{Lie}_F^{\mathrm{pf}}(p)$ mentioned above, the question arises whether $\mathrm{Lie}_F^{\mathrm{pf}}(p^d)$ is always indecomposable.

1.3. To analyze the projective-free part of $\mathrm{Lie}_F(n)$, we are, in particular, interested in the Green vertices and sources of the indecomposable direct summands of $\mathrm{Lie}_F^{\mathrm{pf}}(n)$. Using the reduction result mentioned above, to some extent we are able to reduce this problem for arbitrary n to the case where n is a p-power.