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Coverings of complexes of groups and developability

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ABSTRACT

We develop the theory of coverings of complexes of groups. For $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ a covering of complexes of groups we prove that $\mathcal{G}$ is developable if and only if $\mathcal{G}'$ is developable.

0. Introduction

The Bass–Serre theory of graphs of groups analyzes the algebraic structure of groups acting by automorphisms on simplicial trees. It was formalized by J.P. Serre [9]. The group action on a tree gives a decomposition of this group as a free product with amalgamation or an HNN extension, using the fundamental group of a graph of groups. To every graph of groups $\mathcal{G}$, one can associate a *Bass–Serre covering tree* $\widetilde{X}$, which is a tree that comes equipped with a natural group action of the fundamental group. Moreover, the quotient graph of groups is isomorphic to $\mathcal{G}$. The fundamental theorem of this theory says that if G acts on a tree $\widetilde{X}$ and $\mathcal{G}$ is the associated graph of groups, then G is isomorphic to the fundamental group of $\mathcal{G}$.

The theory of complexes of groups was introduced by M. Bridson and A. Haefliger [1,6] and it provides a higher-dimensional generalization of Bass–Serre theory. A *complex of groups* $\mathcal{G}$ assigns to every object c of a small category $\mathcal{C}$ a group $\mathcal{G}(c)$ and to every morphism $l : c \longrightarrow c'$ a monomorphism of groups $\mathcal{G}(l) : \mathcal{G}(c) \longrightarrow \mathcal{G}(c')$, however this assignment is not necessarily functorial. If a complex of groups admits a good analogue of the notion of the Bass–Serre covering tree, then we say that this complex of groups is *developable*. For example, this is the case if the complex of groups satisfies a non-positive curvature condition, for details cf. Corson [3] and Stallings [10].

The notion of a *covering of a complex of groups* was introduced by Bridson and Haefliger in [1]. It was developed by S. Lim and A. Thomas [7] where they studied the relation between the coverings and developability. Lim and Thomas proved that for a covering of complexes of groups $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$, the developability of $\mathcal{G}$ implies the developability of $\mathcal{G}'$. When it comes to the reverse implication, it was examined

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only for non-positively curved complexes of groups. Precisely, if a complex of groups is non-positively curved, then it is developable, see [1]. Lim and Thomas proved that, for a covering $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$, the complex of groups $\mathcal{G}'$ is non-positively curved if and only if $\mathcal{G}$ is non-positively curved.

The main result of this paper is that the reverse implication holds in general, that is, we prove the following:

Theorem 4.6. *Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ be a covering of complexes of groups. Then $\mathcal{G}$ is developable if and only if $\mathcal{G}'$ is developable.*

0.1. Complex of groups associated to a group action

Assume that a group G acts on a simplicial complex $\widetilde{X}$ in such a way that the orbit space $X := \widetilde{X}/G$ has a natural simplicial structure and the quotient map $q : \widetilde{X} \to X$ is simplicial. Simplices of X are partially ordered by the reverse inclusion; thus they form a category $\mathcal{C}$. We define a "weak" functor from $\mathcal{C}$ to the category of groups $\mathcal{G} : \mathcal{C} \to \mathrm{Gr}$ assigning to every simplex $c \in \mathcal{C}$ first a simplex $\tilde{c} \in q^{-1}(c)$ and then its isotropy subgroup $G_{\tilde{c}}$. If $c' \subset c$ then we pick up an element $g \in G$ such that $\tilde{c}'$ is a face of the simplex $g\tilde{c}$. We define a monomorphism $\psi_{c'c} : G_{\tilde{c}} \to G_{\tilde{c}'}$ as a composition of inclusion $G_{\tilde{c}} \subset G_{g\tilde{c}'}$ and conjugation $\mathrm{Ad}(g) : G_{g\tilde{c}'} \to G_{\tilde{c}'}$. Because of the choices, if we consider the composition $G_{\tilde{c}} \longrightarrow G_{\tilde{c}'} \longrightarrow G_{\tilde{c}''}$ then the monomorphism $\psi_{c''c} \neq \psi_{c''c'}\psi_{c'c}$ and differs from it by the conjugation with an element of the group $G_{\tilde{c}''}$ called the twisting element. Note that $\mathcal{C}$ is a small category such that the only endomorphisms of objects are identities. Such a category is called a small category without loops or *scwol* for short.

These considerations led Haefliger in 1990 to a definition of complexes of groups, that is "weak" functors defined on categories related to simplicial complexes with values in the category of groups and monomorphisms. Much earlier Bob Thomason [12] considered – for homotopy theoretical purpose – similar ideas in much more general categorical context. He considered "weak" functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ (he called them "op-lax functors") from an arbitrary small category to the category of small categories. We note that the definition of Haefliger is a special case of Thomason's when we assume that $\mathcal{C}$ has no loops and the functor takes values in the category of groups and monomorphisms. This is because every group G can be considered as a small category $\mathcal{B}G$ with a single object and the group G as its morphisms.

Each complex of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ associates a small category $\mathcal{B}\mathcal{G}$ called the classifying category of a complex of groups (see [1,6]). It turns out that it is a special case of the Grothendieck construction defined by Thomason. Roughly speaking the classifying category of a complex of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a small category generated by the local groups $\{\mathcal{G}(c)\}_{c \in \mathrm{Ob}\, \mathcal{C}}$ and the small category $\mathcal{C}$. In particular there exists a projection $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$ which is a bijection on the set of objects. If $\mathcal{G}$ is simply a group G then the classifying category of it is $\mathcal{B}G$.

We generalize the latter observation as follows. Suppose a complex of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is associated to an action of a group G on a simply connected simplicial complex $\widetilde{X}$. Then the geometric realization of the classifying category $\mathcal{B}\mathcal{G}$ is homotopy equivalent to the Borel construction $\mathrm{EG} \times_G \widetilde{X}$ where EG is the universal covering of the Eilenberg–MacLane space BG (Theorem 3.2).

1. The categorical background

We begin by recalling some basic notions and constructions used further in the paper.

A small category $\mathcal{C}$ is a category whose morphisms form a set. If c and c' are objects of $\mathcal{C}$ and if l is a morphism of c in c', then c is denoted by $i(l)$ and c' by $t(l)$. Two morphisms l and l' are composable if and