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On Jacobian algebras associated with the once-punctured torus

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A R T I C L E I N F O

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A B S T R A C T

We consider two non-degenerate potentials for the quiver arising from the once-punctured torus, which are a natural choice to study and compare: the first is the Labardini-potential, yielding a finite-dimensional Jacobian algebra, whereas the second potential gives rise to an infinite dimensional Jacobian algebra. In this paper we determine the graph of strongly reduced components for both Jacobian algebras. Our main result is that the graph is connected in both cases. Plamondon parametrised the strongly reduced components for finite-dimensional algebras using generic **g**-vectors. We prove that the generic **g**-vectors of indecomposable strongly reduced components of the finite-dimensional Jacobian algebra are precisely the universal geometric coefficients for the once-punctured torus, which were determined by Reading.

1. Introduction

Cluster algebras were first introduced by Fomin and Zelevinsky in [11] in 2002. One of the main problems in cluster algebra theory is to find a basis of a cluster algebra with favourable properties. It is conjectured [6, Conjecture 1.1] (and similarly before in [15]) that the indecomposable strongly reduced components of Jacobian algebras parametrise such a basis of the corresponding Caldero–Chapoton algebra which sits between the cluster algebra and the upper cluster algebra [6, Proposition 7.1].

Let $A = \mathbb{C}\langle\langle Q\rangle\rangle/I$ be a (possibly infinite dimensional) basic algebra. In [6] the authors generalise the Derksen–Weyman–Zelevinsky E-invariant $E_A(-,?)$ and the notion of strongly reduced components to arbitrary basic algebras. Denote by $\mathrm{decIrr}(A)$ the set of irreducible components of the varieties of finite-dimensional (decorated) representations of A and let $Z_1, \dots, Z_t$ be in $\mathrm{decIrr}(A)$. Then by [6, Theorem 5.11] the Zariski closure $\overline{Z_1 \oplus \cdots \oplus Z_t}$ is a strongly reduced irreducible component if and only if each Z_i is strongly reduced and the generic E-invariant $E_A(Z_i, Z_j) = 0$ for all $i \neq j$.

Denote by $\mathrm{decIrr}^{\mathrm{s.r.}}(A)$ the subset of $\mathrm{decIrr}(A)$ consisting of the strongly reduced components. The graph $\Gamma(A) = \Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(A))$ of strongly reduced components has as vertices the indecomposable components

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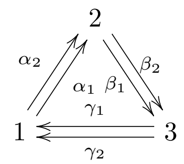
in $\mathrm{decIrr}^{\mathrm{s.r.}}(A)$, and there is an edge between – possibly equal – vertices Z_1 and Z_2 if $E_A(Z_1, Z_2) = E_A(Z_2, Z_1) = 0$. A *component cluster* of A is the set of vertices $\mathcal{U} \subset \Gamma(A)_0$ of a maximal complete subgraph of $\Gamma(A)$. A component cluster $\mathcal{U}$ of A is *E_A-rigid* provided that each $Z \in \mathcal{U}$ is E_A-rigid, i.e. if we have $E_A(Z) = 0$ for all $Z \in \mathcal{U}$. The idea behind these definitions is that the Caldero–Chapoton-functions associated with a component cluster are a generalisation of the clusters of a cluster algebra.

Now, let Q be the quiver



associated with the once-punctured torus, also known as the Markov quiver. There are several aspects which make this quiver an exceptional and much-studied example. For instance, the cluster algebra $\mathcal{A}_Q$ is properly contained in the upper cluster algebra $\mathcal{A}_Q^{\mathrm{up}}$ [3, Proposition 1.26]. Both the cluster algebra $\mathcal{A}_Q$ and the Jacobian algebras arising from Q constitute counterexamples to many conjectures. Still, in particular its Jacobian algebras are not well understood, yet. Consider the two non-degenerate potentials

$$W := \gamma_1\beta_1\alpha_1 + \gamma_2\beta_2\alpha_2$$
$$W' := \gamma_1\beta_1\alpha_1 + \gamma_2\beta_2\alpha_2 - \gamma_2\beta_1\alpha_2\gamma_1\beta_2\alpha_1$$

yielding one infinite dimensional Jacobian algebra $\Lambda = \mathcal{P}(Q, W)$ [8, Example 8.6] and one finite-dimensional Jacobian algebra $\Lambda' = \mathcal{P}(Q, W')$ [17, Example 8.2]. Recently Geuenich [13] proved that there are infinitely many non-degenerate potentials for Q up to right equivalence. Among these, W and W' are a natural choice to study and compare. We study the strongly reduced components of the module varieties of Λ and Λ'. Our main tool is the truncation of basic algebras, which was introduced in [6]. The truncation of Λ allows us to consider Λ as a finite-dimensional string algebra. Thus we can use the purely combinatorial description of the module categories of string algebras. Note that both algebras are tame: Λ is gentle and Λ' is tame by [14, Example 6.5.3].

Our first result is closely connected to τ-tilting theory, which was recently introduced by Adachi, Iyama and Reiten [1], for finite-dimensional basic algebras. They show that τ-tilting theory completes classical tilting theory from the viewpoint of mutation. However, their result does not hold for infinite dimensional algebras in general (see [6, Example 9.3.1]). We show that for Λ the mutation of support τ_Λ^{-1}-tilting modules, which in fact are E_Λ-rigid decorated representations, is always possible and unique.

Proposition 1.1. *If Z_1, Z_2 are indecomposable E_Λ-rigid components in $\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ (resp. in $\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda')$) that are neighbours in $\Gamma := \Gamma(\Lambda)$ (resp. in $\Gamma' := \Gamma(\Lambda')$), then there exist precisely two different indecomposable E_Λ-rigid components $Z_3, Z_3' \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ (resp. in $\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda')$) such that $\{Z_1, Z_2, Z_3\}$ and $\{Z_1, Z_2, Z_3'\}$ are E_Λ-rigid component clusters.*

Our main result is the connectedness of the graph of strongly reduced components for both algebras. It follows from results in [20] that the full subgraph of $\Gamma' = \Gamma(\Lambda')$ on the $E_{\Lambda'}$-rigid components consists of at least two components. Thus the connectedness of Γ' is particularly surprising.

Theorem 1.2.

(i) The graph Γ is connected. The full subgraph on the E_Λ-rigid components is also connected.