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Amoeba basis of zero-dimensional varieties ☆

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A R T I C L E I N F O

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A B S T R A C T

We show that the amoeba of a zero-dimensional variety consisting of a finite number of points has a finite basis. In other words, it is the intersection of finitely many hypersurface amoebas.

1. Introduction

Let K be an algebraically closed field equipped with a non-trivial real valuation $\nu : K \to \mathbb{R} \cup \{\infty\}$. The tropical variety (or non-Archimedean amoeba) $\mathcal{T}rop(\mathcal{I})$ of an ideal $\mathcal{I} \subset K[x_1, \ldots, x_n]$ is defined as the topological closure of the set

$$\nu(V(\mathcal{I})) := \{(\nu(x_1), \ldots, \nu(x_n)) \,|\, (x_1, \ldots, x_n) \in V(\mathcal{I})\} \subset \mathbb{R}^n,$$

where $V(\mathcal{I})$ denotes the zero set of $\mathcal{I}$ in $(K^*)^n$ (see for example [5]). A tropical basis for $\mathcal{I}$ is a generating set $\mathscr{B} = \{g_1, \ldots, g_l\}$ of $\mathcal{I}$ such that

$$\mathcal{T}rop(\mathcal{I}) = \bigcap_{j=1}^{l} \mathcal{T}rop(\mathcal{I}_{g_j}),$$

where $\mathcal{I}_{g_j}$ denotes the principal ideal generated by the polynomial g_j. Bogart, Jensen, Speyer, Sturmfels and Thomas [1] gave a lower bound on the size of such bases when K is the field of Puiseux series $\mathbb{C}((t))$ and the ideal $\mathcal{I}$ is linear with constant coefficients. In [4], Hept and Theobald showed that any tropical variety has a finite tropical basis. The Archimedean amoeba $\mathscr{A}(V)$ of a subvariety V of the complex torus

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$(\mathbb{C}^*)^n$ is its image under the coordinatewise logarithm map. Archimedean amoebas were introduced by Gelfand, Kapranov and Zelevinsky in 1994 [3]. The coamoeba of a subvariety of $(\mathbb{C}^*)^n$ is its image under the coordinatewise argument map to the real torus $(S^1)^n$ (see e.g., [7] for more details about coamoebas). The purpose of this note is to show that Hept and Theobald's theorem has an analogue for Archimedean amoebas of codimension n.

Let V_f denotes the hypersurface with defining polynomial f. If the variety V is generated by the set of polynomials $\{g_i\}_{i=1}^l$ with the following properties:

(i) $\mathscr{A}(V) = \bigcap_{i=1}^l \mathscr{A}(V_{g_i})$;
(ii) $\mathscr{A}(V) \subsetneq \bigcap_{i\in\{1,\dots,l\}\setminus s} \mathscr{A}(V_{g_i})$ for every $1 \le s \le l$,

then we say that $\{g_i\}_{i=1}^l$ is a finite *amoeba basis* of $\mathscr{A}(V)$.

It was shown in [6] that the amoeba of a generic complex algebraic variety of codimension $1 < r < n$ does not have a finite basis. In other words, it is not the intersection of finitely many hypersurface amoebas. The aim of this paper is to show that if the codimension of our variety V is equal to n (i.e., V is a finite number of points), then its amoeba has a finite basis. In other words, the main result of Hept and Theobald in [4] does have an analogue for Archimedean amoebas of 0-dimensional complex algebraic varieties.

Theorem 1.1. *Let $V \subset (\mathbb{C}^*)^n$ be a zero-dimensional variety consisting of a finite number of points. Then the amoeba $\mathscr{A}(V)$ of V has an amoeba basis.*

2. Preliminaries

Let us recall the description of hyperplane amoebas given by Forsberg, Passare and Tsikh [2] and some of their feature that we will use in our proof. In Corollary 4.3 [2], Forsberg, Passare and Tsikh showed the following:

Proposition 2.1 *(Forsberg–Passare–Tsikh). The amoeba $\mathscr{A}(V_f)$ of an affine complex function $f(z) = b_0 + b_1z_1 + \cdots + b_nz_n$ is equal to the closed (possibly empty) subset in $\mathbb{R}^n$ defined by the inequalities*

$$\log|b_0| \le \log(\sum_{k=1}^n |b_k|e^{u_k}),$$

$$u_j + \log|b_j| \le \log(|b_0| + \sum_{k\ne j}^n |b_k|e^{u_k}),$$

for $j = 1,\dots,n$, and where the variables are the u_i's.

By setting $r_k = e^{u_k}$ for $k = 1,\dots,n$, Proposition 2.1 says that the image by the exponential map of the amoeba of an affine complex hyperplane defined by an affine linear complex function $f(z) = b_0 + b_1z_1 + \cdots + b_nz_n$ is equal to the set of points $(r_1,\dots,r_n) \in \mathbb{R}_+^n$ that satisfy the generalized triangle inequalities given by:

$$|b_0| \le \sum_{k=1}^n |b_k|r_k,$$

$$r_j|b_j| \le |b_0| + \sum_{k\ne j}^n |b_k|r_k, \quad \text{for } j = 1,\dots n.$$