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Moduli of abelian covers of elliptic curves

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A R T I C L E I N F O

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A B S T R A C T

For any finite abelian group G, we study the moduli space of abelian G-covers of elliptic curves, in particular identifying the irreducible components of the moduli space. We prove that in the totally ramified case, the moduli space has trivial rational Picard group, and it is birational to the moduli space $M_{1,n}$, where n is the number of branch points. In the particular case of moduli of bielliptic curves, we also prove that the boundary divisors are a basis of the rational Picard group of the admissible covers compactification of the moduli space. Our methods are entirely algebro-geometric.

1. Introduction

1.1. Summary of our results

The purpose of this paper is to study the geometry of moduli spaces of abelian covers of elliptic curves. An abelian cover is a cover which is generically a principal G-bundle for G a certain fixed finite abelian group; from this description abelian covers are a natural generalization of double covers. Our original motivating example is indeed the moduli space of bielliptic curves.

To describe the moduli spaces, we use the powerful language of moduli of stable maps to the classifying stack BG, first discovered in [3] and explained in more detail in [1] by Abramovich, Corti and Vistoli. In general, the moduli spaces that we study here are slightly different from the one naïvely described in the first paragraph, in that the action of G on the covering curve $\tilde{C}$ is fixed, and the branch points are ordered (moreover, we also allow ourselves to mark and order other points of the quotient $\tilde{C}/G$). These moduli spaces admit a forgetful map, which only remembers the covering curve, to closed subspaces of the moduli spaces of curves.

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For example, in the case of the moduli space of bielliptic curves, our results are naturally presented for moduli whose objects parameterize tuples $(\tilde{C}, \alpha, x_1, \ldots, x_n)$, where $\tilde{C}$ is a genus-g curve, α is a distinguished involution such that $\tilde{C}/\langle\alpha\rangle$ is a genus-1 curve and $\{x_1, \ldots, x_n\}$ is the branch locus of the map $\tilde{C} \to \tilde{C}/\langle\alpha\rangle$. By quotienting this moduli space by the action of the symmetric group $\mathfrak{S}_n$ on the points x_i, we retrieve the moduli of bielliptic curves without an ordering of the branch points. Moreover, this moduli space admits a finite map to the moduli space where the involution is only required to exist and it is not fixed (for genus $g \geq 6$ there is at most a unique bielliptic involution, so such a finite map is in fact an isomorphism), and the latter has codimension-$(g-1)$ in $\mathcal{M}_g$.

We now summarize the main results of our work. In Theorem 1, we give a geometric description of the irreducible components of the moduli spaces of abelian G-covers of elliptic curves. From this description, we are able to deduce further results for those components that parameterize *totally ramified* covers. In fact, in Section 3.2, we prove that each component of the moduli space parameterizing totally ramified abelian G-covers is birational to $\mathcal{M}_{1,n}$ where n is the number of marked points (in general, a superset of the set of branch points). In Theorem 3, we exploit our description to prove the vanishing of the rational Picard group of such components of the moduli space, and to find the number of independent relations among the boundary divisors of the admissible covers compactification. In the last section we make our results more explicit in the case of the rational Picard group of the moduli spaces of admissible bielliptic curves, see Corollaries 6 and 7.

The Picard number of the compact moduli of hyperelliptic curves $\overline{\mathcal{H}}_g$ is g [20]. In Corollary 6, we see that, when $g > 2$, it is $2g$ for the compact moduli $\overline{\mathcal{B}}_g$ of bielliptic curves (without ordering the branch). A simple analysis of the number of boundary divisors of the compact moduli of double covers of genus-h curves leads us to conjecture that the Picard number of the latter moduli space grows as $(h+1)g + g \cdot o(1)$ for $g \to \infty$.

1.2. Literature and the state of the art

Our moduli spaces are generalizations of moduli of hyperelliptic curves. The geometry of the latter has been thoroughly investigated; we know that they are rational [28], and we have a description of their Picard group [20,6]. We address the reader to Section 2.4, in which we review some known results, relevant for this paper, regarding the geometry of moduli of abelian covers of genus-0 curves and their compactification.

There is a large body of work concerning the birational geometry of moduli of bielliptic curves due to Bardelli, Casnati and Del Centina; see [18] and the references therein. The rational Picard group of moduli of bielliptic curves was investigated by the author in the cases of $g = 2$ [31] and $g = 3$ [32] (with Tommasi). In [23] (with Faber), we investigate the problem of enumerating bielliptic curves of genus 3. The geometry of the Hurwitz scheme of covers of positive genus curves is studied in [25], and the moduli spaces that we are studying are closed subspaces of these Hurwitz schemes, of increasingly high codimension as the order of G increases.

As we are treating the case of curves with trivial canonical bundle, when G is cyclic the objects we are considering are twisted higher level Prym or Spin curves of genus 1. The literature on such moduli spaces and on the problem of compactifying them is vast; we address the reader to the introduction of [15] and the references therein. Our main results are similar to those obtained by Bini and Fontanari [12,13]; we remark however that there is no intersection between our results and theirs, as their interest is focused on the component of the moduli spaces that generically parameterizes étale double covers. Let us remark furthermore that our results are entirely algebro-geometric and hold more generally for $\operatorname{char} k \nmid |G|$.

Our approach of treating moduli of decorated elliptic curves by means of moduli of stable maps to a stack is similar in flavour to the works [34] and [30]; our motivation and main results are orthogonal to theirs.

The results proven in this paper have a non-trivial intersection with a recent paper by Poma, Talpo and Tonini. In [35], they compute the integral Picard group of the stack of uniform G-covers, without