Journal of Pure and Applied Algebra 219 (2015) 3847–3861

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Some results on the vanishing conjecture of differential operators with constant coefficients ☆

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A R T I C L E   I N F O

Article history:
Received 7 June 2014
Received in revised form 15 December 2014
Available online 14 January 2015
Communicated by E.M. Friedlander

MSC:
33C45; 32W99; 14R15

A B S T R A C T

In this paper we prove four cases of the *Vanishing Conjecture* of differential operators with constant coefficients and also a conjecture on the Laurent polynomials with no holomorphic parts, which were proposed in [15] by the third named author. We also give two examples to show that both the *Vanishing Conjecture* and the *Duistermaat–van der Kallen Theorem* [6] cannot be generalized to the setting of (Laurent) formal power series in general.

1. Introduction

Let $z = (z_1, z_2, \dots, z_n)$ be n commutative free variables and $\mathbb{C}[z]$ (resp. $\mathbb{C}[z^{-1}, z]$) the algebra of polynomials (resp. Laurent polynomials) in z over $\mathbb{C}$. For any $1 \le i \le n$, set $\partial_i := \partial/\partial z_i$, and $\partial := (\partial_1, \partial_2, \dots, \partial_n)$.

We say a differential operator Λ of $\mathbb{C}[z]$ is a *differential operator with constant coefficients* if $\Lambda = h(\partial)$ for some polynomial $h(\xi) \in \mathbb{C}[\xi]$, where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ are another n commutative variables which also commute with z. For convenience, we will denote the polynomial $h(\partial)$ by $\Lambda(\partial)$ and simply write $\Lambda = \Lambda(\partial)$.

In this paper, we will prove four cases of the following *Vanishing Conjecture* of differential operators with constant coefficients, which was proposed by the third named author in [15].

Conjecture 1.1. *Let $P(z) \in \mathbb{C}[z]$ and $\Lambda = \Lambda(\partial)$ for some $\Lambda(\xi) \in \mathbb{C}[\xi]$. Assume that $\Lambda^m(P^m) = 0$ for all $m \ge 1$. Then $\Lambda^m(P^{m+1}) = 0$ when $m \gg 0$.*

Actually, all the cases of the conjecture above that we will prove in this paper also hold in the following more general form.

☆ The second named author is supported by the Netherlands Organization of Scientific Research (NWO). The third named author has been partially supported by NSA Grant R1-07-0053.

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http://dx.doi.org/10.1016/j.jpaa.2014.12.024
0022-4049/

Conjecture 1.2. *Let $P(z) \in \mathbb{C}[z]$ and $\Lambda = \Lambda(\partial)$ for some $\Lambda(\xi) \in \mathbb{C}[\xi]$. Assume that $\Lambda^m(P^m) = 0$ for all $m \geq 1$. Then for any $g(z) \in \mathbb{C}[z]$, we have $\Lambda^m(P^m g) = 0$ when $m \gg 0$.*

Note that Conjecture 1.1 is just the special case of Conjecture 1.2 with $g(z) = P(z)$. Also, when Λ is a homogeneous differential operator of order 2 with constant coefficients, these two conjectures are actually equivalent (see [8] and [15]).

Note also that Conjecture 1.2 has recently been generalized by the third named author [16] to the so-called *Image Conjecture* of commuting differential operators of order one with constant leading coefficients. Actually, Conjecture 1.2 is equivalent to the *Image Conjecture* of the commuting differential operators $\xi_i - \partial_i$ ($1 \leq i \leq n$) of the polynomial algebra $\mathbb{C}[\xi, z]$ for the separable polynomial $\Lambda(\xi)P(z) \in \mathbb{C}[\xi, z]$. For more details, see [16].

The main motivation behind Conjecture 1.1 is its connection with the well-known *Jacobian Conjecture* proposed by O.H. Keller [9] in 1939 (see also [1] and [7]). The connection is given by the following theorem proved in [14].

Theorem 1.3. *Let $\Delta_n := \sum_{i=1}^n \partial_i^2$ be the Laplace operator of the polynomial algebra $\mathbb{C}[z]$. Then the Jacobian Conjecture holds for all $n \geq 1$ iff Conjecture 1.1 with $\Lambda = \Delta_n$ holds for all $n \geq 1$.*

It has also been shown in [15] that one may replace the Laplace operators Δ_n in the theorem above by any sequence $\Lambda_n = \Lambda_n(\partial)$ ($n \geq 1$) of differential operators with $\Lambda_n(\xi)$ homogeneous of degree 2, whose ranks go to ∞ as $n \to \infty$.

The proof of Theorem 1.3 is based on the remarkable symmetric reduction achieved independently by M. de Bondt and the first named author [2] and G. Meng [12]. It also depends on some results obtained in [13] on a deformation of polynomial maps.

Currently, there are only a few cases of Conjecture 1.1 that are known. The best results so far come from M. de Bondt and the first named author's results [3,4] on symmetric polynomial maps via the equivalence obtained by the third named author in [14] and [15]. The results state that Conjecture 1.1 holds for homogeneous quadratic differential operators $\Lambda = \Lambda(\partial)$ if either $n \leq 4$ (for all $P(z) \in \mathbb{C}[z]$) or $n \leq 5$ with $P(z)$ homogeneous. The case when $\Lambda(\xi)$ is an (integral) power of a homogeneous linear polynomial of ξ is an easy exercise (see also [15]).

In this paper we will prove four more cases of Conjecture 1.2 and also a conjecture proposed in [15] by the third named author on the Laurent polynomials with no holomorphic parts (see Theorem 5.1).

First, in Section 2 we use a fundamental theorem, Theorem 2.1, in ODE to show that Conjecture 1.2 holds for the one variable case (see Theorem 2.3). Actually, in this case Conjecture 1.2 even holds for all formal power series $P(z)$ and polynomials $g(z) \in \mathbb{C}[z]$.

In Section 3 we assume $n = 2$ and show in Theorem 3.1 that Conjecture 1.2 holds for any differential operator Λ of the form $\partial_x - \Phi(\partial_y)$, where, for convenience, in this section we use (x, y) instead of (z_1, z_2) to denote two free commutative variables and $\Phi(\cdot)$ to denote any polynomial in one variable. We also give an example, Example 3.5, to show that both Conjectures 1.1 and 1.2 cannot be generalized to formal power series in general.

In Section 4 we first recall and prove some results on rational polytopes, polytopes with all its vertices having rational coordinates. We then use the remarkable Duistermaat–van der Kallen theorem [6] (see Theorem 4.5) to show what we refer as the *Density Theorem* of polytopes of Laurent polynomials (see Theorem 4.6). We will also show in Lemma 4.9 that Conjecture 1.2 holds when the polytope $\mathrm{Poly}(P) - \mathrm{Poly}(\Lambda)$ has no intersection points with $(\mathbb{R}^{\geq 0})^{\times n}$. At the end of this section, we give an example, Example 4.11, to show that the *Duistermaat–van der Kallen Theorem* [6] cannot be generalized to the setting of Laurent formal power series.