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Toric ideals associated with gap-free graphs

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ABSTRACT

In this paper we prove that every toric ideal associated with a gap-free graph G has a squarefree lexicographic initial ideal. Moreover, in the particular case when the complementary graph of G is chordal (i.e. when the edge ideal of G has a linear resolution), we show that there exists a reduced Gröbner basis $\mathcal{G}$ of the toric ideal of G such that all the monomials in the support of $\mathcal{G}$ are squarefree. Finally, we show (using work by Herzog and Hibi) that if I is a monomial ideal generated in degree 2, then I has a linear resolution if and only if all powers of I have linear quotients, thus extending a result by Herzog, Hibi and Zheng.

1. Introduction

Algebraic objects depending on combinatorial data have attracted a lot of interest among both algebraists and combinatorialists: some valuable sources to learn about this research area are the books by Stanley [24], Villarreal [27], Miller and Sturmfels [12], and Herzog and Hibi [7]. It is often a challenge to establish relationships between algebraic and combinatorial properties of these objects.

Let G be a simple graph and consider its vertices as variables of a polynomial ring over a field K. We can associate with each edge e of G the squarefree monomial M_e of degree 2 obtained by multiplying the variables corresponding to the vertices of the edge. With this correspondence in mind, we can now introduce some algebraic objects associated with the graph G:

- the *edge ideal* $I(G)$ is the monomial ideal generated by $\{M_e \mid e \text{ is an edge of } G\}$;
- the *toric ideal* I_G is the kernel of the presentation of the K-algebra $K[G]$ generated by $\{M_e \mid e \text{ is an edge of } G\}$.

An important result by Fröberg [5] gives a combinatorial characterisation of those graphs G whose edge ideal $I(G)$ admits a linear resolution: they are exactly the ones whose complementary graph G^c is chordal.

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Another strong connection between the realms of commutative algebra and combinatorics is the one which links initial ideals of the toric ideal I_G to triangulations of the edge polytope of G, see Sturmfels's book [25] and the recent article by Haase, Paffenholz, Piechnik and Santos [6]. Furthermore, Gröbner bases of I_G have been studied among others by Ohsugi and Hibi [21] and Tatakis and Thoma [26]. A necessary condition for I_G to have a squarefree initial ideal is the normality of $K[G]$, which was characterised combinatorially by Ohsugi and Hibi [19] and Simis, Vasconcelos and Villarreal [23]. Normality, though, is not sufficient: Ohsugi and Hibi [16] gave an example of a graph G such that $K[G]$ is normal but all possible initial ideals of I_G are not squarefree.

An interesting class of graphs is the one consisting of the so-called *gap-free graphs* (following Dao, Huneke and Schweig's notation in [3]), i.e. graphs such that any two edges with no vertices in common are linked by at least one edge. Unfortunately, these graphs do not have a standard name in the literature. Just to name a few possibilities:

- graph theorists refer to gap-free graphs as "$2K_2$-free graphs" and so do Hibi, Nishiyama, Ohsugi and Shikama in [9];
- Nevo and Peeva call them "C_4-free graphs" in [13] and [14];
- Ohsugi and Hibi use the phrase "graphs whose complement is weakly chordal" in [18];
- Corso and Nagel call bipartite gap-free graphs "Ferrers graphs" in [2].

The main goal of this paper is to prove that the toric ideal I_G has a squarefree lexicographic initial ideal, provided the graph G is gap-free (Theorem 3.9): moreover, the corresponding reduced Gröbner basis consists of circuits. In the particular case when $I(G)$ has a linear resolution (Theorem 3.6) we are actually able to prove that the reduced Gröbner basis $\mathcal{G}$ we describe consists of circuits such that all monomials (both leading and trailing) in the support of $\mathcal{G}$ are squarefree, thus extending a result of Ohsugi and Hibi [17] on multipartite complete graphs.

In [8] Herzog, Hibi and Zheng proved that the following conditions are equivalent:

(a) $I(G)$ has a linear resolution;
(b) $I(G)$ has linear quotients;
(c) $I(G)^k$ has a linear resolution for all $k \geq 1$.

It is quite natural to ask (see for instance the article by Hoefel and Whieldon [11]) whether these conditions are in turn equivalent to the fact that

(d) $I(G)^k$ has linear quotients for all $k \geq 1$.

In Theorem 2.6 we prove that this is indeed the case, as can be deduced from results in [7]. Note that all the equivalences between conditions (a), (b), (c), (d) above hold more generally for monomial ideals generated in degree 2 which are not necessarily squarefree.

The computer algebra system CoCoA [1] gave us the chance of performing computations which helped us to produce conjectures about the behaviour of the objects studied.

2. Notation and known facts

First of all, let us fix some notation. K will always be a field and G a simple graph with vertices $V(G) = \{1, \ldots, n\}$ and edges $E(G) = \{e_1, \ldots, e_m\}$. We can associate with each edge $e = \{i, j\}$ the degree 2 monomial (called *edge monomial*) $M_e := x_i x_j \in K[x_1, \ldots, x_n]$ and hence we can consider the *edge ideal*