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A generalization of Wantzel's Theorem, m-sectable angles, and the density of certain Chebyshev-polynomial images

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A B S T R A C T

The eponymous theorem of P.L. Wantzel [5] presents a necessary and sufficient criterion for angle trisectability in terms of the third Chebyshev polynomial T_3, thus making it easy to prove that there exist non-trisectable angles. We generalize this theorem to the case of all Chebyshev polynomials T_m (Corollary 1.4.1). We also study the set **m-Sect** consisting of all cosines of m-sectable angles (see Section 1), showing that, when m is not a power of two, **m-Sect** contains only algebraic numbers (Theorem 1.1). We then introduce a notion of density based on the diophantine-geometric concept of height of an algebraic number and obtain a result on the density of certain polynomial images. Using this in conjunction with the Generalized Wantzel Theorem, we obtain our main result: for every real algebraic number field K, the set **m-Sect** $\cap K$ has density zero in $[-1, 1] \cap K$ when m is not a power of two (Corollary 1.5.1).

1. Introduction

This paper poses and answers some interesting algebraic questions raised by P.L. Wantzel's 1837 theorem that destroyed the age-old hope of finding a "ruler and compass" construction for angle-trisection.[1] More precisely, Wantzel [5] proved the following result:

Wantzel's Theorem. *Let α be any angle, and set $\cos(\alpha) = a$.[2] Then α admits a trisection using only an unmarked straightedge and compass if and only if the polynomial $4x^3 - 3x - a$ has a zero in the field $\mathbb{Q}(a)$.*

It is easy to see, as Wantzel did, that when $4x^3 - 3x - a$ satisfies the algebraic criterion of his theorem, the number a must be algebraic.[3] Thus, many (in fact, most) angles are not trisectable.

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[1] See [1], pp. 2, 3, for a thumbnail sketch of the history of this problem leading to Wantzel's work.

[2] We continue with this notation throughout this introduction.

[3] The converse, however, is false. For example, there are infinitely many non-trisectable angles whose cosines are rational numbers. E.g., see Lemma 2.4(b) below, or see [1], p. 8 ff.

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Here are four questions suggested by Wantzel's Theorem.

The first question involves extending or generalizing the theorem. Let m be any positive integer. We say that α is m-sectable if it admits an m-fold equipartition by a construction that uses only an unmarked straightedge and compass. When $m = 2$ (resp., $m = 3$) we use the familiar terms bisectable (resp., trisectable) instead. For a given m, we say that *m-sectability always holds* if every angle is m-sectable. Otherwise we say that *m-sectability sometimes fails*. Wantzel's Theorem shows that trisectability sometimes fails. We can now ask the following:

(A) *Can we extend Wantzel's Theorem to the case of m-sectability, for $m > 3$?*

This question requires some preliminary discussion, which we defer. Instead we ask an easier question:

(B) *Suppose α is m-sectable for some m. Must $a = \cos(\alpha)$ be an algebraic number?*

This has a fairly easy, direct answer:

Theorem 1.1. *If m is a power of two, then m-sectability always holds. In other words, the quantity a can assume any value in the unit interval $[-1, 1]$. However, when m is not a power of two, α is m-sectable only if a is an algebraic number in $[-1, 1]$.*

The first sentence of the theorem is obviously true, since bisectability always holds. It is mentioned only for completeness.

We denote the field of algebraic numbers by $\overline{\mathbb{Q}}$, and we let **m-Sect** denote the set of cosines of m-sectable angles. By Theorem 1.1, when m is not a power of two, we have an inclusion of countable sets

$$\textbf{m-Sect} \subseteq \overline{\mathbb{Q}} \cap [-1, 1].$$

We now ask:

(C) *When m is not a power of two, how densely is* **m-Sect** *distributed in $\overline{\mathbb{Q}} \cap [-1, 1]$?*

The notion of density that we use is tied to the concept of *height* of an algebraic number; we describe this briefly in Section 3. A comprehensive discussion of height may be found in [2]. Here, we say only that, given an algebraic number field K, there is a function $H_K : K \to [1, \infty)$, called the *height function on K*, with the important property that, for every real number $B \in [1, \infty)$, the set $H_K^{-1}[1, B]$ is non-empty and finite.

We consider sets $S \subseteq T \subseteq \mathbb{C}$ such that $1 \in T$. We then define the *K-density of S in T* to be the limit as $B \to \infty$ of the quotients of finite cardinalities

$$\delta_K(S, T; B) = \frac{|S \cap H_K^{-1}[1, B]|}{|T \cap H_K^{-1}[1, B]|}, \tag{1}$$

provided this limit exists. We denote the limit by $\delta_K(S, T)$.

Theorem 1.2. *Let K be an algebraic number field $\subset \mathbb{R}$, and let m be any positive integer. Then, the K-density $\delta_K(\textbf{m-Sect}, [-1, 1])$ exists. It equals 1 when m is a power of two, and it equals 0 otherwise.*

The first assertion in the last sentence is immediate from Theorem 1.1. The second is a consequence of Corollary 1.5.1 below. An analogue of this, in which the infinite-dimensional field $\overline{\mathbb{Q}}$ replaces K, is still out of reach. If true, it seems to require numerical estimates more delicate than those used here (in Section 3). See the remark after Proposition 1.5 and Corollary 1.5.1.

We now return to question **(A)**. The polynomial $4x^3 - 3x$ appearing in Wantzel's Theorem will no doubt be recognized by many as the *third Chebyshev polynomial* $T_3(x)$. Its connection with angle trisectability is