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The primitive element theorem for differential fields with zero derivation on the ground field

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ABSTRACT

In this paper we strengthen Kolchin's theorem [1] in the ordinary case. It states that if a differential field E is finitely generated over a differential subfield $F \subset E$, $\operatorname{trdeg}_F E < \infty$, and F contains a nonconstant, i.e., an element f such that $f' \neq 0$, then there exists $a \in E$ such that E is generated by a and F. We replace the last condition with the existence of a nonconstant element in E.

1. Introduction

All fields considered in this paper are of characteristic zero. Let us briefly recall some basic notions of differential algebra. Let R be a ring. A map $D: R \to R$ satisfying $D(a+b) = D(a) + D(b)$ and $D(ab) = aD(b)+D(a)b$ for all $a, b \in R$ is called *derivation*. We will denote $D(x)$ by x' and $D^n(x)$ by $x^{(n)}$. A *differential ring* R is a ring with a specified derivation. A differential ring which is a field will be called a *differential field*. Let $F \subset E$ be a differential field extension and $a \in E$. Let us denote by $F\langle a\rangle$ the differential subfield of E generated by F and a. If $F\langle a\rangle = E$, then element a is said to be *primitive*. An element $a \in R$ of the differential ring R is said to be *constant* if $a' = 0$.

Kolchin proved [1] a differential analogue of the primitive element theorem:

Theorem. *(See [1], p. 728, case $m = 1$.) Let $E = F\langle a_1, \ldots, a_n\rangle$ and $\operatorname{trdeg}_F E < \infty$. Assume also that F contains a nonconstant element. Then, there exists $b \in E$ such that $E = F\langle b\rangle$.*

Remark 1. In [1] Kolchin considered a more general case, i.e., fields equipped with a set of m commuting derivations. He required the existence of m elements in F whose Jacobian is nonzero. In this paper we restrict ourselves to the ordinary case.

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The following is an easy consequence of the above theorem.

Corollary. *Let $E = F\langle a_1, \dots, a_n\rangle$ and $\operatorname{trdeg}_F E < \infty$. Assume also that E contains a nonconstant element. Then, there exist $b, c \in E$ such that $E = F\langle b, c\rangle$.*

The last condition of the above theorem cannot be omitted. Indeed, let us consider the field $E = \mathbb{Q}(x, y)$ equipped with zero derivation. Clearly, the extension $F = \mathbb{Q} \subset E$ has no primitive element.

Kolchin's theorem was extended to positive characteristic by Seidenberg in [2]. Finally, both cases of zero and positive characteristic were summarized by Kolchin in his book [3] in terms of differential separability [3, Chap. 2, Prop. 9]. In [4] Babakhanian constructed primitive elements for several specific extensions $F \subset E$, namely when E is logarithmic differential field.

The goal of the present paper is to prove the primitive element theorem for the case when E contains a nonconstant element and $f' = 0$ for all $f \in F$. Results of this kind seem to be applicable, for example, to the study of computable differential fields and constrained extensions (see [5, Th. 4.7] and [5, Th. 8.6]).

2. Main results

Throughout the rest of the paper F denotes a field equipped with the trivial derivation.

Theorem 1. *Let $E = F\langle a, b\rangle$, $\operatorname{trdeg}_F E < \infty$, and $b' \neq 0$. Then, there exists $p(x) \in \mathbb{Q}[x]$ such that $\operatorname{trdeg}_F F\langle a + p(b)\rangle = \operatorname{trdeg}_F F\langle a, b\rangle$.*

Theorem 2. *Let $E = F\langle a_1, \dots, a_m\rangle$, $\operatorname{trdeg}_F E < \infty$, and E contains a nonconstant. Then, there exists $a \in E$ such that $E = F\langle a\rangle$.*

Remark 2. Unlike Kolchin's proof it is not sufficient to consider elements of the form $a + \lambda b$ ($\lambda \in F$). For example, let $\mathbb{Q}(x, y)$ be a differential field with the derivation defined by $x' = 1$ and $y' = 0$. There is no primitive element of the form $y + \lambda x$ ($\lambda \in \mathbb{Q}$), but $\mathbb{Q}(x, y) = \mathbb{Q}\langle x^2 + y\rangle$.

Proof of Theorem 1. We will need the following well-known lemmas:

Lemma 1. *If $\operatorname{trdeg}_F F\langle a\rangle = n$, then $F\langle a\rangle = F\left(a, a', \dots, a^{(n)}\right)$.*

Proof. Let m be the minimal integer such that $a, \dots, a^{(m)}$ are algebraically dependent over F. Let $R(x_1, \dots, x_m)$ be a polynomial over F of minimal degree in x_m such that $R(a, \dots, a^{(m)}) = 0$. Hence

$$0 = \left(R(a, \dots, a^{(m)})\right)' = \sum_{i=0}^{m} a^{(i+1)} \frac{\partial}{\partial x_i} R(a, \dots, a^{(m)})$$

Thus, $a^{(m+1)} \in F(a, \dots, a^{(m)})$.

Similarly we obtain that $a^{(N)} \in F(a, \dots, a^{(m)})$ for all $N > m$. Hence, $n = m$ and $F\langle a\rangle = F\left(a, \dots, a^{(n)}\right)$. $\square$

Lemma 2. *(See [6, p. 35].) Let $q(x, x', \dots, x^{(n)})$ be a nonzero differential polynomial over a differential field E. Let $f \in E$ be a nonconstant element. Then, there exists $p(t) \in \mathbb{Q}[t]$ such that*

$$q(x, x', \dots, x^{(n)})\Big|_{x=p(f)} \neq 0$$