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Relative singularity categories

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ABSTRACT

We study the properties of the relative derived category $D^b_{\mathscr{C}}(\mathscr{A})$ of an abelian category $\mathscr{A}$ relative to a full and additive subcategory $\mathscr{C}$. In particular, when $\mathscr{A} = A$-mod for a finite-dimensional algebra A over a field and $\mathscr{C}$ is a contravariantly finite subcategory of A-mod which is admissible and closed under direct summands, the $\mathscr{C}$-singularity category $D_{\mathscr{C}\text{-sg}}(\mathscr{A}) = D^b_{\mathscr{C}}(\mathscr{A})/K^b(\mathscr{C})$ is studied. We give a sufficient condition when this category is triangulated equivalent to the stable category of the Gorenstein category $\mathscr{G}(\mathscr{C})$ of $\mathscr{C}$.

1. Introduction

Let A be a finite-dimensional algebra over a field. We denote by A-mod the category of finitely generated left A-modules, and A-proj (resp. A-inj) the full subcategory of A-mod consisting of projective (resp. injective) modules. We use $K^b(A)$ and $D^b(A)$ to denote the bounded homotopy and derived categories of A-mod respectively, and $K^b(A\text{-proj})$ (resp. $K^b(A\text{-inj})$) to denote the bounded homotopy category of A-proj (resp. A-inj).

The composition functor $K^b(A\text{-proj}) \to K^b(A) \to D^b(A)$ with the former one the inclusion functor and the latter one the quotient functor is naturally a fully faithful triangle functor, and then one can view $K^b(A\text{-proj})$ as a triangulated subcategory of $D^b(A)$. In fact it is a thick one by [7, Lemma 1.2.1]. Consider the quotient triangulated category $D_{sg}(A) := D^b(A)/K^b(A\text{-proj})$, which is the so-called "singularity category". This category was first introduced and studied by Buchweitz in [7] where A is assumed to be a left and right noetherian ring. Later on Rickard proved in [26] that for a self-injective algebra A, this category is triangle-equivalent to the stable category of A-mod. This result was generalized to Gorenstein algebra by Happel in [19]. Since A has finite global dimension if and only if $D_{sg}(A) = 0$, from this viewpoint $D_{sg}(A)$ measures the homological singularity of the algebra A, we call it the singularity category after [24].

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Besides, other quotient triangulated categories have been studied by many authors. Beligiannis considered the quotient triangulated categories $D^b(R\text{-Mod})/K^b(R\text{-Proj})$ and $D^b(R\text{-Mod})/K^b(R\text{-Inj})$ for arbitrary ring R, where R-Mod is the category of left R-modules and R-Proj (resp. R-Inj) is the full subcategory of R-Mod consisting of projective (resp. injective) modules (see [5]). Let $\mathscr{A}$ be an abelian category. A full and additive subcategory ω of $\mathscr{A}$ is called *self-orthogonal* if $\mathrm{Ext}^i_{\mathscr{A}}(M,N)=0$ for any $M,N\in\omega$ and $i\geq 1$; in particular, an object T in $\mathscr{A}$ is called *self-orthogonal* if $\mathrm{Ext}^i_{\mathscr{A}}(T,T)=0$ for any $i\geq 1$. Chen and Zhang studied in [11] the quotient triangulated category $D^b(A)/K^b(\mathrm{add}_A T)$ for a finite-dimensional algebra A and a self-orthogonal module T in A-mod, where $\mathrm{add}_A T$ is the full subcategory of A-mod consisting of direct summands of finite direct sums of T. Recently Chen studied in [10] the relative singularity category $D_\omega(\mathscr{A}):=D^b(\mathscr{A})/K^b(\omega)$ for an arbitrary abelian category $\mathscr{A}$ and an arbitrary self-orthogonal, full and additive subcategory ω of $\mathscr{A}$.

For an abelian category $\mathscr{A}$ with enough projective objects, the Gorenstein derived category $D^*_{gp}(\mathscr{A})$ of $\mathscr{A}$ was introduced by Gao and Zhang in [16], where $*\in\{\text{blank},-,b\}$. It can be viewed as a generalization of the usual derived category $D^*(\mathscr{A})$ by using Gorenstein projective objects instead of projective objects and $\mathscr{GP}$-quasi-isomorphisms instead of quasi-isomorphisms, where $\mathscr{GP}$ means "Gorenstein projective". For Gorenstein projective modules and Gorenstein projective objects, we refer to [2,13,14,20,27]. Asadollahi, Hafezi and Vahed studied in [1] the relative derived category $D^*_{\mathscr{C}}(\mathscr{A})$ for an arbitrary abelian category $\mathscr{A}$ with respect to a contravariantly finite subcategory $\mathscr{C}$, where $*\in\{\text{blank},-,b\}$, and they pointed out that $K^b(\mathscr{C})$ can be viewed as a triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$.

Given a finite-dimensional algebra A over a field and a full and additive subcategory $\mathscr{C}$ of $\mathscr{A}(=A\text{-mod})$ closed under direct summands, it follows from [6] that $K^b(\mathscr{C})$ is a Krull–Schmidt category and hence can be viewed as a thick triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$. If the quotient triangulated category $D_{\mathscr{C}\text{-}sg}(\mathscr{A}):=D^b_{\mathscr{C}}(\mathscr{A})/K^b(\mathscr{C})$ is considered, then it is natural to ask whether $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ shares some nice properties of $D_{sg}(A)$. The aim of this paper is to study this question.

In Section 2, we give some terminology and some preliminary results.

In Section 3, for an abelian category $\mathscr{A}$ and a full and additive subcategory $\mathscr{C}$ of $\mathscr{A}$, we prove that if $\mathscr{C}$ is admissible, then the composition functor $\mathscr{A}\to K^b(\mathscr{A})\to D^b_{\mathscr{C}}(\mathscr{A})$ is fully faithful, where the former functor is the inclusion functor and the latter one is the quotient functor. Let $\mathscr{C}$ be a contravariantly finite subcategory of $\mathscr{A}$ and $\mathscr{D}\subseteq\mathscr{A}$ a subclass of $\mathscr{A}$. We introduce a dimension denoted by $\mathscr{C}\mathscr{D}\text{-}\dim M$ which is called the *$\mathscr{C}$-proper $\mathscr{D}$-dimension* of an object M in $\mathscr{A}$. By choosing a left $\mathscr{C}$-resolution $C^\bullet_M$ of M, we get a functor $\mathrm{Ext}^n_{\mathscr{C}}(M,-):=H^n\mathrm{Hom}_{\mathscr{A}}(C^\bullet_M,-)$ for any $n\in\mathbb{Z}$. Then by using the properties of this functor we obtain some equivalent characterizations for $\mathscr{C}\mathscr{C}\text{-}\dim M$ being finite.

In Section 4, we introduce the $\mathscr{C}$-singularity category $D_{\mathscr{C}\text{-}sg}(\mathscr{A}):=D^b_{\mathscr{C}}(\mathscr{A})/K^b(\mathscr{C})$, where $\mathscr{A}=A$-mod and $\mathscr{C}$ is a contravariantly finite, full and additive subcategory of $\mathscr{A}$ which is admissible and closed under direct summands. We prove that if $\mathscr{C}\mathscr{C}\text{-}\dim\mathscr{A}<\infty$, then $D_{\mathscr{C}\text{-}sg}(\mathscr{A})=0$. As a consequence, we get that if A is of finite representation type, then $\mathscr{C}\mathscr{C}\text{-}\dim\mathscr{A}<\infty$ if and only if $D_{\mathscr{C}\text{-}sg}(\mathscr{A})=0$. Let $\mathscr{G}(\mathscr{C})$ be the Gorenstein category of $\mathscr{C}$ and ε the collection of all $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact complexes of the form: $0\to L\to M\to N\to 0$ with $L,M,N\in\mathscr{G}(\mathscr{C})$. By [8] (or [25]) $(\mathscr{G}(\mathscr{C}),\varepsilon)$ is an exact category; moreover, it is a Frobenius category with $\mathscr{C}$ the subcategory of projective–injective objects, see [18]. We prove that if $\mathscr{C}\mathscr{G}(\mathscr{C})\text{-}\dim\mathscr{A}<\infty$, then the natural functor $\theta:\mathscr{G}(\mathscr{C})\to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ induces a triangle-equivalence $\theta':\underline{\mathscr{G}(\mathscr{C})}\to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$, where $\underline{\mathscr{G}(\mathscr{C})}$ is the stable category of $\mathscr{G}(\mathscr{C})$.

2. Preliminaries

Throughout this paper, $\mathscr{A}$ is an abelian category, $C(\mathscr{A})$ is the category of complexes of objects in $\mathscr{A}$, $K^*(\mathscr{A})$ is the homotopy category of $\mathscr{A}$ and $D^*(\mathscr{A})$ is the usual derived category by inverting the quasi-isomorphisms in $K^*(\mathscr{A})$, where $*\in\{\text{blank},-,b\}$. We will use the formula $\mathrm{Hom}_{K(\mathscr{A})}(X^\bullet,Y^\bullet[n])=H^n\mathrm{Hom}_{\mathscr{A}}(X^\bullet,Y^\bullet)$ for any $X^\bullet,Y^\bullet\in C(\mathscr{A})$ and $n\in\mathbb{Z}$ (the ring of integers).