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Invariant deformation theory of affine schemes with reductive group action

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ABSTRACT

We develop an invariant deformation theory, in a form accessible to practice, for affine schemes W equipped with an action of a reductive algebraic group G. Given the defining equations of a G-invariant subscheme $X \subset W$, we device an algorithm to compute the universal deformation of X in terms of generators and relations up to a given order. In many situations, our algorithm even computes an algebraization of the universal deformation. As an application, we determine new families of examples of the invariant Hilbert scheme of Alexeev and Brion, where G is a classical group acting on a classical representation, and we describe their singularities.

1. Introduction

Let k be a fixed algebraically closed field of characteristic zero. Let us fix a reductive algebraic group G, an affine G-scheme W of finite type over k, and a Hilbert function $h : \mathrm{Irr}(G) \to \mathbb{N}$ which assigns to every irreducible representation of G a nonnegative integer. We denote by $\mathcal{H} := \mathrm{Hilb}_h^G(W)$ the invariant Hilbert scheme of Alexeev and Brion [1] corresponding to the triple (G, W, h); see Section 2.1 for more details.

This is a quite thrilling and somewhat mysterious object, and there is a large number of articles dedicated to its study. It has rendered services to the classification theory of spherical varieties; see [7, Section 4] for an overview and further references. Moreover, in many cases the invariant Hilbert scheme furnishes a canonical candidate for a resolution of singularities of the categorical quotient $W//G = \mathrm{Spec}(k[W]^G)$. Indeed, if we take $h = h_0$ the Hilbert function of the general fibers of the quotient morphism $W \to W//G$ (see Section 2.1 for the precise definition of h_0), then we have the so-called *Hilbert–Chow morphism* $\gamma : \mathcal{H} \to W//G$, which is a projective morphism that maps a unique irreducible component $\mathcal{H}^{\mathrm{main}}$ of $\mathcal{H}$, the so-called *main component*,

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birationally to $W//G$. Examples where γ is a resolution can be found in [15,16,5,19] for finite groups, and in [18,2,28,29] for classical groups.

On the other side, it is difficult to get control about the invariant Hilbert scheme in a hands-on way. It has been described only in some very special situations where $\mathcal{H}$ was generally first shown to be smooth by some ad hoc arguments. Examples where $h = h_0$ can be found in the references mentioned above, and examples where h takes values in $\{0, 1\}$ can be found in [17,4,21,8]. However, when $\mathcal{H}$ is singular, explicit descriptions of examples as well as a general strategy to describe the singularities were missing so far.

The goal of this article is to describe an algorithm which provides Zariski-local equations of $\mathcal{H}$ at an arbitrary point; see Section 5 and in particular Algorithm 5.1. This is, in some sense, the strongest form of information that one can have about a scheme. To achieve this, we developed an invariant deformation theory in a form accessible to practice; see Section 3. Our algorithm is completely general and can be applied to any point $[X] \in \mathcal{H}$ as soon as there is an action of the multiplicative group $\mathbb{G}_m$ on W by G-equivariant automorphisms with strictly positive weights on $k[W]$ and on the cotangent space $(T_{[X]}\mathcal{H})^\vee$; see Hypothesis 4.1.

As an illustration, we apply our algorithm in three situations:

(1) $G = GL_3$ acting on $W = (k^3)^{\oplus n_1} \oplus (k^{3*})^{\oplus n_2}$, which is the sum of n_1 copies of the defining representation, and n_2 copies of its dual (Section 7.3);
(2) $G = SO_3$ acting on $W = (k^3)^{\oplus 3}$ (Section 6.1); and
(3) $G = O_3$ acting on $W = (k^3)^{\oplus n}$ (Section 7.1).

Theorems 6.1 and 7.3. *Let G and W be as in situation 1 or 2, and let $h = h_0$ be the Hilbert function of the general fibers of the quotient morphism $\nu : W \to W//G$. Then the main component of the invariant Hilbert scheme is smooth, and thus the Hilbert–Chow morphism $\gamma : \mathcal{H}^{\mathrm{main}} \to W//G$ is a resolution of singularities. In both cases, $\mathcal{H}$ is reduced, connected, and the union of two irreducible components $\mathcal{H}^{\mathrm{main}} \cup \mathcal{H}'$, where $\mathcal{H}'$ is smooth in situation 1 but singular in situation 2.*

Moreover, we give a description of the special fiber $\gamma^{-1}(\nu(0))$, both as an abstract scheme as well as in terms of the G-stable ideals that it parametrizes. Even in the case where we do not succeed to describe the invariant Hilbert scheme completely, our algorithm proves helpful. We obtain the following information in situation 3:

Theorem 7.1. *Let G and W be as in situation 3, and let $h = h_0$ be the Hilbert function of the general fibers of the quotient morphism $W \to W//G$. Then the invariant Hilbert scheme $\mathcal{H}$ is connected and has at least two irreducible components.*

We give more detailed formulations of these results in Sections 6 and 7. These examples have entered the focus by the work [27] because they were the first examples with classical groups acting on classical representations where the invariant Hilbert scheme was known to be singular. However, this was shown simply by calculating the dimension of the tangent space and comparing it to the dimension of $W//G$. Thus, no further properties of $\mathcal{H}$ such as reducibility, or the smoothness of the main component $\mathcal{H}^{\mathrm{main}}$, were given. Our results also show that the geometry of the invariant Hilbert scheme can be very diverse. Thus, it is important to have many more examples, and our algorithm gives a powerful tool to calculate them.

Let us explain how we obtain these results. The strategy is to *localize* geometric properties of $\mathcal{H}$ in special points and then to compute local equations at these points via Algorithm 5.1. The question is: which points in $\mathcal{H}$ contain most information and how do we find them? In situations 1 to 3, there is each time an algebraic subgroup $H \subset \mathrm{Aut}^G(W)$ acting on $\mathcal{H}$. Imagine that we want to study the singular locus $\mathcal{H}^{\mathrm{Sing}} \subset \mathcal{H}$ for