Journal of Pure and Applied Algebra 220 (2016) 1681–1706

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Twisted modules for toroidal vertex algebras

Fei Kong [a], Haisheng Li [b,1], Shaobin Tan [a,2], Qing Wang [a,*,3]

[a] School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
[b] Department of Mathematical Sciences, Rutgers University, Camden, NJ 08102, USA

ARTICLE INFO

Article history:
Received 12 March 2015
Received in revised form 6 August 2015
Available online 26 September 2015
Communicated by C.A. Weibel

ABSTRACT

This is a paper in a series systematically to study toroidal vertex algebras and their (twisted) modules. Previously, a theory of toroidal vertex algebras and modules was developed and toroidal vertex algebras were explicitly associated to toroidal Lie algebras. In this paper, we study twisted modules for toroidal vertex algebras. More specifically, we introduce a notion of twisted module for a general toroidal vertex algebra with a finite order automorphism and we give a general construction of toroidal vertex algebras and twisted modules. We then use this construction to establish a natural association of toroidal vertex algebras and twisted modules to twisted toroidal Lie algebras. This together with some other known results implies that most of the extended affine Lie algebras can be associated to toroidal vertex algebras.

1. Introduction

Extended affine Lie algebras are natural generalizations of affine Kac–Moody Lie algebras (see [1]), where an important family of extended affine Lie algebras consists of toroidal Lie algebras which are central extensions of multi-loop algebras of finite dimensional simple Lie algebras. It is known that (untwisted) affine Lie algebras can be canonically associated with vertex algebras and modules (cf. [8,6,12,14]), while twisted affine Lie algebras can be associated with vertex algebras in terms of twisted modules (see [7,13]). In [3], a natural connection of certain toroidal Lie algebras with vertex algebras was established. On the other hand, it is natural to study suitable toroidal analogues of vertex algebras and their relations with toroidal Lie algebras from a different perspective. This is also potentially important from the viewpoint of conformal field theory in high dimensions in physics (cf. [9,10]). With this as the main driving force, a theory

* Corresponding author.
E-mail address: qingwang@xmu.edu.cn (Q. Wang).

[1] Partially supported by China NSFC grant (No. 11471268).
[2] Partially supported by China NSFC grant (Nos. 11471268, 11531004).
[3] Partially supported by China NSFC grant (Nos. 11371024, 11531004), Natural Science Foundation of Fujian Province (No. 2013J01018) and Fundamental Research Funds for the Central University (No. 2013121001).

http://dx.doi.org/10.1016/j.jpaa.2015.09.002
0022-4049/

of toroidal vertex algebras and modules was developed and toroidal vertex algebras and their modules were associated to toroidal Lie algebras in [16].

In the current paper, we continue to study twisted modules for toroidal vertex algebras, with a goal to associate twisted modules for certain toroidal vertex algebras to modules for twisted toroidal Lie algebras. To achieve this goal, we develop a theory of twisted modules for a toroidal vertex algebra with a finite order automorphism and we establish a conceptual construction of toroidal vertex algebras and twisted modules. By using this general result, we successfully associate twisted modules for certain toroidal vertex algebras to twisted toroidal Lie algebras.

Note that affine Kac–Moody algebras were classified as untwisted affine Lie algebras and twisted affine Lie algebras, where untwisted affine Lie algebras can be realized as the universal central extensions of loop algebras of finite dimensional simple Lie algebras and twisted affine Lie algebras can be realized as fixed points subalgebras of untwisted affine Lie algebras under Dynkin digram automorphisms (see [11]). For extended affine Lie algebras, essentially this is also the case; it was proved (see [2,17,18,4,5,19]) that all extended affine Lie algebras except those constructed from the centerless irrational Lie tori can be realized as twisted toroidal Lie algebras. In view of this, almost all extended affine Lie algebras can be associated to toroidal vertex algebras.

Now, we give a more detailed account of the contents of this paper. First, an $(r+1)$-toroidal vertex algebra (with r a positive integer) is defined (see [16]) to be a vector space V equipped with a linear map

$$\begin{aligned} Y(\cdot;x_0,\mathbf{x}): \quad & V \to \mathrm{Hom}(V, V[[x_1^{\pm 1},\dots,x_r^{\pm 1}]]((x_0))) \\ & v \mapsto Y(v;x_0,\mathbf{x}), \end{aligned}$$

where $\mathbf{x}=(x_1,\dots,x_r)$, and equipped with a distinguished vector $\mathbf{1}$ of V, such that

$$Y(\mathbf{1};x_0,\mathbf{x})v = v \quad \text{and} \quad Y(v;x_0,\mathbf{x})\mathbf{1} \in V[[x_0,x_1^{\pm 1},\dots,x_r^{\pm 1}]] \quad \text{for } v \in V,$$

and such that for $u,v \in V$,

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{x_0-y_0}{z_0}\right)Y(u;x_0,\mathbf{zy})Y(v;y_0,\mathbf{y}) - z_0^{-1}\delta\left(\frac{y_0-x_0}{-z_0}\right)Y(v;y_0,\mathbf{y})Y(u;x_0,\mathbf{zy}) \\ & = y_0^{-1}\delta\left(\frac{x_0-z_0}{y_0}\right)Y(Y(u;z_0,\mathbf{z})v;y_0,\mathbf{y}), \end{aligned}$$

where

$$Y(u;x_0,\mathbf{zy}) = \sum_{(m_0,\mathbf{m})\in\mathbb{Z}\times\mathbb{Z}^r} u_{m_0,\mathbf{m}} x_0^{-m_0-1}\mathbf{z}^{-\mathbf{m}}\mathbf{y}^{-\mathbf{m}}.$$

Now, let V be an $(r+1)$-toroidal vertex algebra with a finite order automorphism σ of period N. We define a *σ-twisted V-module* to be a vector space W equipped with a linear map

$$\begin{aligned} Y_W(\cdot;x_0,\mathbf{x}): \quad & V \to \mathrm{Hom}(W, W[[x_1^{\pm 1},\dots,x_r^{\pm 1}]]((x_0^{\frac{1}{N}}))) \\ & v \mapsto Y_W(v;x_0,\mathbf{x}) \end{aligned}$$

such that

$$Y_W(\mathbf{1};x_0,\mathbf{x}) = 1_W,$$

and for $u,v \in V$,