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Commutativity

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ABSTRACT

We describe a general framework for notions of commutativity based on enriched category theory. We extend Eilenberg and Kelly's tensor product for categories enriched over a symmetric monoidal base to a tensor product for categories enriched over a normal duoidal category; using this, we re-find notions such as the commutativity of a finitary algebraic theory or a strong monad, the commuting tensor product of two theories, and the Boardman–Vogt tensor product of symmetric operads.

1. Introduction

This article is a category-theoretic investigation into the notion of *commutativity*. We first meet commutativity in elementary algebra: two elements a, b of a monoid M are said to commute if $ab = ba$, while M itself is called commutative if all its elements commute pairwise. This immediately yields other notions of commutativity: for groups (on forgetting the inverses), for rings (on forgetting the additive structure) and for Lie algebras (on passing to the universal enveloping algebra).

Later on, we encounter more sophisticated forms of commutativity not directly reducible to that for monoids. For example, a pair of operations f, g of arities m, n in an algebraic theory $\mathcal{T}$ are said to *commute* if the two mn-ary operations

$$
\begin{aligned}
& f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn})) \\
\text{and} \qquad & g(f(x_{11}, \dots, x_{m1}), \dots, f(x_{1n}, \dots, x_{mn}))
\end{aligned}
$$

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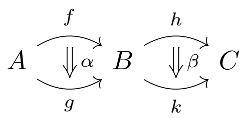
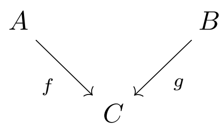
are equal, while $\mathcal{T}$ itself is *commutative* when all of its operations commute pairwise; typical commutative theories are those for join-semilattices, for commutative monoids and for modules over a commutative ring R. An important related notion in this context is the *commuting tensor product* $\mathcal{S} \odot \mathcal{T}$ of theories $\mathcal{S}$ and $\mathcal{T}$; this has the property that $\mathcal{S} \odot \mathcal{T}$-models in a category $\mathcal{E}$ correspond with $\mathcal{S}$-models in the category of $\mathcal{T}$-models in $\mathcal{E}$, and also with $\mathcal{T}$-models in the category of $\mathcal{S}$-models in $\mathcal{E}$. There is a corresponding notion of commutativity for operations in *symmetric operads* in the sense of [42], and the analogue of the commuting tensor product in this context is the *Boardman–Vogt* tensor product of [6].

Yet another kind of generalised commutativity arises in the context of the *sesquicategories* of [48]; these may be defined succinctly as comprising a category $\mathcal{C}$ together with a lifting of $\mathbf{Hom}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ through the forgetful functor $\mathbf{Cat} \to \mathbf{Set}$. To give such a lifting is to equip $\mathcal{C}$ with 2-cells that admit vertical composition and whiskering on each side with 1-cells, but which need not satisfy the *interchange* axiom, which requires that for any pair of 2-cells in the configuration



we should have $\beta g \circ h\alpha = k\alpha \circ \beta f \colon hf \Rightarrow kg$. If we declare the pair (α, β) to commute just when they *do* satisfy interchange, then a sesquicategory will be commutative, in the sense of all of its composable pairs commuting, precisely when it is a 2-category. A related example involves the *premonoidal categories* of [46], which bear the same relation to (non-strict) monoidal categories as sesquicategories do to 2-categories.

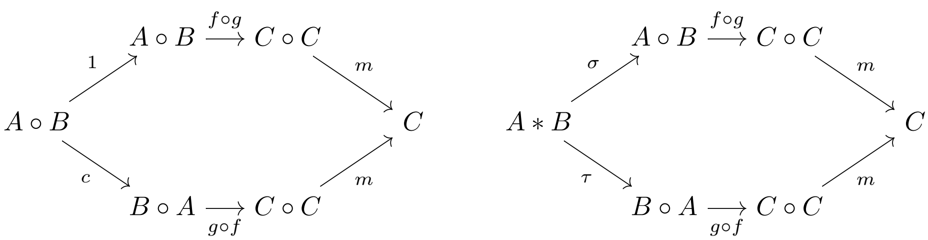
The objective of this paper is to describe an abstract framework for commutativity that encompasses each of the examples given above, and others besides. As a starting point, we observe that each of our examples is concerned with a kind of structure—monoids, algebraic theories, operads, sesquicategories—that can be viewed as a monoid in a particular monoidal category; an ordinary monoid, is, of course, a monoid in the cartesian monoidal $\mathbf{Set}$, while a finitary algebraic theory can be seen as a monoid in the substitution monoidal category $[\mathbb{F}, \mathbf{Set}]$, where $\mathbb{F}$ is the category of functions between finite cardinals. Consequently, the key notion of our abstract theory will be the definition, for a suitable monoidal category $(\mathcal{V}, \circ, I)$ and a monoid C therein, of what it means for a pair of generalised elements



(1.1)

of C to commute. As explained in [27], other aspects of the theory flow easily once this definition is made: for example, C itself is commutative just when the generalised element 1_C commutes with itself, while the commuting tensor product of monoids A and B is the universal monoid $A \odot B$ in which A and B commute; other notions such as centralizers and centres also admit expression in this generality.

When $(\mathcal{V}, \circ, I)$ is a *braided* monoidal category, it is easy to say when (1.1) should be a commuting cospan—namely, just when the left-hand diagram in:



(1.2)