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Orbit closures and rational surfaces

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ABSTRACT

In this paper we study the Grassmannian of submodules of a given dimension inside the radical of a finitely generated projective module P for a finite dimensional algebra Λ over an algebraically closed field. The orbit of such a submodule C under the action of $\mathrm{Aut}_\Lambda(P)$ on the Grassmannian encodes information on the top-stable degenerations of P/C. The goal of this article is to begin the study of the global geometry of the closures of such orbits. In dimension one, this geometry is determined by the local rings of singular points. The smallest dimension for which the global geometry is not determined by local data is two, and this case is our main focus. We give several examples to illustrate the interplay between the geometry of the projective surfaces which arise and the corresponding posets of top-stable degenerations.

1. Introduction

Let Λ be a basic finite dimensional algebra over an algebraically closed field k. A fundamental problem in representation theory is the classification of families of finitely generated Λ-modules. Gabriel and Kac approached this problem by studying the affine scheme $\mathbf{Mod}_d(\Lambda)$ that parameterizes the left Λ-modules with fixed dimension d [8,9,15,16]. In particular, this led to considering the closures of the orbits of the action under conjugation by GL_d on $\mathbf{Mod}_d(\Lambda)$, as in the work of Bongartz, Kraft, Riedtmann, Schofield, Skowronski, Zwara and many others (see for example [3,4,18,19,21–25]). If M and M' are two Λ-modules

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corresponding to points in $\mathbf{Mod}_d(\Lambda)$, then M' is said to be a degeneration of M if M' lies in the closure of the GL_d-orbit of M. By [21,25], the degenerations of M correspond to the Λ-modules M' for which there exists a Λ-module X together with an exact sequence

$$0 \to X \to X \oplus M \to M' \to 0.$$

In spite of this useful algebraic characterization, the partially ordered set of degenerations of M is far from understood in general.

In this paper, we shift the focus to an alternate, projective, parameter variety, namely to the Grassmannian of submodules of a given dimension inside the radical of a finitely generated projective Λ-module P, as in the work of Bongartz and the third author [6,7,12–14]. The orbit $\mathrm{Aut}_\Lambda(P).C$ of such a submodule C under the action of $\mathrm{Aut}_\Lambda(P)$ is the set of submodules C' of P such that P/C and P/C' are isomorphic. Points C' in the closure $\overline{\mathrm{Aut}_\Lambda(P).C}$ of this orbit correspond to top-stable degenerations of P/C. In other words, these are the degenerations of P/C which have the same top (i.e., the same radical quotient) as P/C. Letting T be the top of P/C, one can also define top-stable degenerations in the affine setting. Namely, one uses the locally closed subscheme $\mathbf{Mod}_d^T$ of $\mathbf{Mod}_d(\Lambda)$ such that the closed points of $\mathbf{Mod}_d^T$ correspond to d-dimensional Λ-modules M whose top is isomorphic to T (see [12, Sect. 2]).

The purpose of this paper is to initiate an investigation of the global geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$. Our primary motivation is to shed more light on the poset of top-stable degenerations of P/C. We will focus on the case when P is an indecomposable projective module.

To be more specific regarding applications, one rationale for our study lies in the fact that the global geometry of the orbit closure holds the key to a classification of the degenerations P/C' of P/C in situations (such as the one we are addressing) where full control is a realistic goal. In particular, this geometry provides access to the stratification of the boundary of $\overline{\mathrm{Aut}_\Lambda(P).C}$ in terms of the "heights" of its points. This partition, in turn, forms the basis for a conjectural classification up to isomorphism of the top-stable degenerations of P/C (see [13]). Furthermore, the geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$ clearly governs the complexity of families of modules which "continuously" bridge the gap between a given boundary point of the orbit and interior points. More precisely, by work of Kollár in [17] (see also [13, Prop. 4.2]), each boundary point C' belongs to the closure of a rational curve of sufficiently high degree in $\mathrm{Aut}_\Lambda(P).C$. A natural question is how high this degree must be. We will show how answers may be derived from information regarding the global geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$.

To place our investigation into the context of existing work, local questions about orbit closures, such as types of singularities occurring over specific algebras and regularity questions have been explored by several authors (see, e.g., [2,5,26]). Our present goal, namely to move from the local to the global geometry, is more comprehensive, but a full grasp is likely to remain limited to a narrower range of cases. On the other hand, an analysis of situations where the global geometry is within reach furnishes prototypes. This aspect is extensively illustrated in the final section of the paper, where we exhibit hierarchies of degenerations of P/C in tandem with relatively minimal smooth projective models of the minimal desingularization of $\overline{\mathrm{Aut}_\Lambda(P).C}$. This provides examples of different ways of subdividing classes of modules in terms of "shapes" of posets of top-stable degenerations.

In dimension one, the global geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$ is determined by the local rings of singular points, since the minimal desingularization of $\overline{\mathrm{Aut}_\Lambda(P).C}$ is isomorphic to $\mathbb{P}_k^1$. The first case in which the global geometry is not determined by local data is in dimension two, and this case is our main focus. As in the classification of algebraic varieties, the natural approach is to proceed by increasing dimension, and the case of surfaces is crucial for understanding varieties of higher dimension.

When $\mathrm{Aut}_\Lambda(P).C$ has dimension two and P is indecomposable, we will show that the global geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$ can be bounded by $\dim_k(C)$ in the following sense.