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Cellular objects and Shelah's singular compactness theorem

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ABSTRACT

The best-known version of Shelah's celebrated singular cardinal compactness theorem states that if the cardinality of an abelian group is singular, and all its subgroups of lesser cardinality are free, then the group itself is free. The proof can be adapted to cover a number of analogous situations in the setting of non-abelian groups, modules, graph colorings, set transversals, etc. We give a single, structural statement of singular compactness that covers all examples in the literature that we are aware of. A case of this formulation, singular compactness for cellular structures, is of special interest; it expresses a relative notion of freeness. The proof of our functorial formulation is motivated by a paper of Hodges, based on a talk of Shelah. The cellular formulation is new, and related to recent work in abstract homotopy theory.

0. Introduction

The form of Shelah's celebrated Singular Compactness Theorem that is the oldest historically and prototypical in its formulation is the following: if μ is a singular cardinal and A is an abelian group of size μ all of whose subgroups of cardinality less than μ are free, then A itself is free. In his breakthrough work [17], however, Shelah already proved variants of singular compactness for algebras other than abelian groups, as well as for certain graph colorings and set transversals, to which Hodges [8] and Eklof and Mekler [5] added numerous other examples. These examples are unified, roughly speaking, by the fact that the proof works for them. Both Shelah [17] and Hodges [8] (which is also based on Shelah's ideas) contain axiomatizations for the proofs to carry through. The *intrinsic* meaning of singular compactness remains slightly mysterious. The best formulation that seems to be available is the informal "if μ is a singular cardinal and S a structure

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all of whose substructures of cardinality less than μ are free, then S itself is free." Cf. the title of Hodges [8] and the first sentence of Eklof [4].

The goal of this article, essentially, is to make the above sentence precise: to formulate singular compactness by specifying how *structure* and *free* should be understood in this context. The obvious guess, that structure should mean 'first-order definable structure,' turns out to be too broad; and the guess that free should mean 'free algebra on a set,' turns out to be too restrictive.

Our starting point is the straightforward translation of the prototypical case of singular compactness into the setting of category theory. Consider the free abelian group functor $F : Set \to Ab$. An abelian group is free if and only if it is isomorphic to an object in the image of this functor. Our desired form of the singular compactness theorem is: for suitable functors $F : \mathcal{A} \to \mathcal{B}$, if the size of an object X of $\mathcal{B}$ is singular, and every subobject of X of lesser size is in the image of F, then X itself is in the image of F. (For the sake of brevity, we will write 'is in the image of the functor' to mean 'is isomorphic to an object in the image of the functor'.)

A category is *accessible* if it is equivalent to the category of models and homomorphisms of a set of sentences of the form $\phi \to \psi$, where ϕ and ψ belong to the positive-existential fragment of the logic $L_{\kappa,\kappa}$ for some κ. This class of categories was identified by Makkai and Paré [11] as having the right mix of properties to develop a categorical model theory of infinitary logics. There exist language-independent characterizations of accessible categories, using only concepts of category theory. One needs to assume that $\mathcal{A}$ is an accessible category with directed colimits, $\mathcal{B}$ is a finitely accessible category (this is a sub-class of accessible categories with directed colimits, corresponding, roughly, to $\kappa = \omega$ in the above definition) and the functor F preserves directed colimits. These assumptions allow one to introduce a notion of *size* of objects that is determined purely by the ambient category; see Definition 1.4. The assumptions on directed colimits are indispensable for creating transfinite chains of subobjects. The final assumption is that *F-structures extend along morphisms*. This is a simple diagrammatic condition; see Definition 1.1. Thinking of F as a 'free' functor and $\mathcal{A}$ as a category of 'bases' whose morphisms are 'extensions of bases', the condition expresses the matroid-like property that any partial basis (i.e. independent set) can be extended to a basis.

Singular compactness theorem (functorial form): Let $\mathcal{A}$ be an accessible category with filtered colimits, $\mathcal{B}$ a finitely accessible category and $F : \mathcal{A} \to \mathcal{B}$ a functor preserving filtered colimits. Assume that F-structures extend along morphisms. Let $X \in \mathcal{B}$ be an object whose size μ is a singular cardinal. If all subobjects of X of size less than μ are in the image of F, then X itself is in the image of F.

The formulation given in the paper, Theorem 1.6, is slightly more general in that it only assumes that 'enough' subobjects of X lie in the image of F. The actual criterion, using dense filters of subobjects, was inspired by the treatment of singular compactness in Eklof and Mekler [5].

Returning to the paradigmatic example of singular compactness, there is another way to think of free abelian groups or more generally, free algebras. Define a class F_α of algebras, α ranging over the ordinals, and compatible morphisms $F_\beta \to F_\alpha$ for $\beta \prec \alpha$, by transfinite induction. Let $F_\varnothing$ be $F(\varnothing)$, the free algebra on an empty set, and let $F_\bullet$ be the free algebra on a singleton. For successor $\alpha + 1$, let $F_{\alpha+1}$ be the pushout

$$\begin{array}{ccc} F_\varnothing & \longrightarrow & F_\alpha \\ \downarrow & & \downarrow \\ F_\bullet & \longrightarrow & F_{\alpha+1} \end{array}$$

For limit α, let F_α be $\operatorname{colim}_{\beta \prec \alpha} F_\beta$. Then an algebra is free if and only if it is isomorphic to F_α for some α.

The above diagram is really a coproduct, since $F_\varnothing$ is the initial object of Alg. Working in an arbitrary cocomplete category and allowing an arbitrary member of a fixed collection I of morphisms to be pushed on at successor ordinals, the process being continuous at limit ordinals, one obtains the important class of