Journal of Pure and Applied Algebra 220 (2016) 2001–2016

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Symbolic powers of planar point configurations II

M. Dumnicki [a], T. Szemberg [b,*], H. Tutaj-Gasińska [a]

[a] *Jagiellonian University, Faculty of Mathematics and Computer Science, Łojasiewicza 6, PL-30348 Kraków, Poland*
[b] *Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland*

A R T I C L E I N F O

Article history:
Received 28 April 2015
Received in revised form 6 September 2015
Available online 26 October 2015
Communicated by A.V. Geramita

MSC:
14C20; 14J26; 14N20; 13A15; 13F20

A B S T R A C T

In [12] we began to study the initial sequences $\alpha(I^{(m)})$, $m = 1, 2, 3 \ldots$, of radical ideals I of finite sets of points in the projective plane. In the present note we complete results obtained in [12] by answering a number of questions left open in the previous note and we extend our considerations to the asymptotic setting of Waldschmidt constants. The concept of the Bezout decomposition introduced in Definition 2.7 might be of independent interest.

1. Introduction

Symbolic powers of ideals of point configurations have attracted considerable attention in recent years. Apart from traditional paths of research motivated by various problems in Algebraic Geometry, Commutative Algebra and Combinatorics (see e.g. [7,18,20]), ideals of planar points have been recently studied in connection with the counterexamples to the $I^{(3)} \subset I^2$ containment (see e.g. [13,6,11,17]) and with the Bounded Negativity Conjecture (see [2]). These recent directions of investigation focus on special configurations of points. The study of such configurations, from a yet slightly different point of view, has been initiated by Bocci and Chiantini in [5]. We follow their approach in the present note.

For a homogeneous ideal $I = \bigoplus_{d \geqslant 0} I_d$, we define the *initial degree* $\alpha(I)$ of I as the least integer d such that $I_d \neq 0$. More generally, we define the *initial sequence* (or simply the α-sequence of I) as the strictly increasing sequence of integers

$$\alpha(I) < \alpha(I^{(2)}) < \alpha(I^{(3)}) < \alpha(I^{(4)}) < \ldots, \quad (1)$$

where $I^{(m)}$ denotes the m-th symbolic power of I.

* Corresponding author.
E-mail addresses: Marcin.Dumnicki@im.uj.edu.pl (M. Dumnicki), tomasz.szemberg@gmail.com (T. Szemberg), Halszka.Tutaj@im.uj.edu.pl (H. Tutaj-Gasińska).

http://dx.doi.org/10.1016/j.jpaa.2015.10.012
0022-4049/

There is a related asymptotic quantity introduced by Chudnovsky [8] and rediscovered by Harbourne who named it the *Waldschmidt constant* of I:

$$\widehat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m} = \inf_{m\geqslant 1} \frac{\alpha(I^{(m)})}{m}.$$

For radical ideals $I = I(Z)$ attached to configurations Z of points in $\mathbb{P}^2$ Bocci and Chiantini studied the question to what extent the value of the difference $\alpha(I^{(2)}) - \alpha(I)$ determines the geometry of Z. Their results, still for planar points configurations, have been considerably generalized in [12] and further extended to other varieties in [19,1,3,10]. Many ideas presented here can be adapted to a more general setting. We don't dwell on this point in order to keep the presentation as transparent as possible.

It is convenient to define the *first differences sequence* of the initial sequence as $\beta_m(I) = \alpha(I^{(m+1)}) - \alpha(I^{(m)})$ for $m \geqslant 1$ and to set $\beta_0(I) = \alpha(I)$. We call this sequence simply the β-sequence of I. Of course, the α sequence determines the β sequence and vice versa. In the present note we focus on zero sets of ideals whose β-sequence contains many twos and threes.

Our main result is the following classification statement, see Definition 2.2 for the terminology applied.

Main Theorem. *Let $Z = \{P_1, \ldots, P_s\}$ be a finite set of points in $\mathbb{P}^2$ and let $I = I(Z)$ be its radical ideal. If*

$$\widehat{\alpha}(I) < \frac{9}{4},$$

then Z

a) is contained in a line ($\widehat{\alpha}(I) = 1$) or a conic ($\widehat{\alpha}(I) \leqslant 2$) or;
b) is a 4-star ($\widehat{\alpha}(Z) = 2$) and $s = 6$ in this case.

Moreover if $\alpha(I^{(m)}) = 9/4$ for some m, then Z is a 3-quasi star.

As sample consequences of the Main Theorem we derive the following results.

Corollary A. *Let I be the radical ideal of a finite set Z of points in $\mathbb{P}^2$.*

a) If there exists $m \geqslant 1$ such that $\beta_m(I) = \beta_{m+1}(I) = 1$, then $\alpha(I) = 1$, i.e. the set Z is contained in a line.
b) If there exists $m \geqslant 1$ such that $\beta_m(I) = \beta_{m+1}(I) = \beta_{m+2}(I) = \beta_{m+3}(I) = \beta_{m+4}(I) = 2$, then $\alpha(I) = 2$ i.e. Z is contained in a conic.
c) For any $d \geqslant 3$ there exist configurations of points such that $\beta_m(I) = d$ for all $m \geqslant 2$ but $\alpha(I) \geqslant d + 1$.

Corollary B. *Let I be the radical ideal of a finite set Z of points in $\mathbb{P}^2$ with an integral Waldschmidt constant $\widehat{\alpha}(I) = d$.*

(i) If $d = 1$, then $\alpha(I) = 1$, i.e. Z is contained in a line.
(ii) If $d \geqslant 2$, then $\alpha(I)$ need not be equal to d, i.e. there exist configurations of points Z with $\alpha(I) \geqslant d + 1$.

We work over an algebraically closed field of characteristic 0.

2. Preliminaries and auxiliary results

In this section we fix the notation and recall very useful results of Chudnovsky and Esnault and Viehweg.