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Equivariant formality of isotropy actions on homogeneous spaces defined by Lie group automorphisms

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A B S T R A C T

Given a compact, connected Lie group G and a compact, connected Lie subgroup K defined by an automorphism of G, we show that the isotropy action of K on G/K is equivariantly formal and that (G, K) is a Cartan pair.

1. Introduction

An action of a compact Lie group K on a manifold M is called equivariantly formal if the equivariant cohomology $H^*_K(M)$ associated with that action is a free module over the polynomial ring $S^*(\mathfrak{k}^*)^{\mathrm{Ad}_K}$. Although it might not be immediately evident, this condition has strong topological consequences, the most important one being that it makes the equivariant cohomology algebra more accessible to explicit computations: e.g., the equivariant cohomology of an equivariantly formal torus action embeds, by Borel's Localization Theorem, into the equivariant cohomology of the fixed point set. Results of this type demonstrate why it is desirable to be able to identify equivariantly formal actions.

A particular class of manifolds that come equipped with two canonical Lie group actions is that of homogeneous spaces G/K. The actions just referred to are induced by multiplication with elements of K or G from the left, and the former is known as the *isotropy action of K on G/K*. The question of whether or not the isotropy action is equivariantly formal has been investigated by a variety of authors, under different assumptions on G and K, but mostly under the premise that both G and K are at least compact. For example, the answer to this question is always affirmative if G and K are of equal rank or if G/K is a symmetric space of the compact type, cf. [6] and [4], respectively; on the other hand, the answer might be negative if K is a circular subgroup of G, see [3].

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Here, we provide yet another class of homogeneous spaces for which the isotropy action always is equivariantly formal, namely, the class comprised of those spaces which are defined by Lie group automorphisms.

Theorem 1.1. *Let K be a closed, connected Lie subgroup of a compact, connected Lie group G. If the fixed point subalgebra of an automorphism of G coincides with the Lie algebra of K, then the pair (G,K) is a Cartan pair and the isotropy action of K on G/K is equivariantly formal.*

Theorem 1.1 uses several facts about equivariant formality and the cohomology of compact homogeneous spaces, so a brief summary of the relevant statements and notions, such as that of a Cartan pair (Definition 2.4), is provided in Section 2.

Among the examples to which Theorem 1.1 applies are the homogeneous spaces defined by automorphisms of finite order, commonly referred to as *generalized symmetric spaces*. As the name suggests, these include the symmetric spaces, but also non-symmetric examples such as the 3-symmetric space $\mathrm{Spin}(8)/G_2$ induced by the triality automorphism, the 12-symmetric space $\mathrm{Spin}(8)/T^2$ considered in [11, Theorem 12] or the 3-symmetric space $\mathrm{Spin}(8)/SU(3)$ which appears in [16, Table 3]. While for this particular class the property of being formal in the sense of Sullivan — which, for a homogeneous space G/K, is equivalent to (G,K) being a Cartan pair — has been verified by various authors, e.g. in [11] and [14], there have been, to our knowledge, only partial results on the equivariant formality of isotropy actions of generalized symmetric spaces which are not symmetric. For instance, the results of [1] imply that the isotropy action of $\mathrm{Spin}(8)/G_2$ is equivariantly formal.

Verifying Theorem 1.1 for generalized symmetric spaces is essential for its proof because the automorphisms of simple Lie algebras can be built by composing inner automorphisms and automorphisms of finite order, the latter of which have been classified by V. Kač (see [9, Section X.5]). In the first step of the proof we, therefore, focus on pairs (G,K) which satisfy the assumptions of Theorem 1.1 and for which G is furthermore assumed to be simple. This step is the content of Section 4. Using arguments from Section 3, we observe that the statement of Theorem 1.1 is true for such pairs of Lie groups if it can be established for those pairs of Lie groups that may arise from automorphisms of the Dynkin diagrams; by invoking, ultimately, the classification of the automorphisms of the Dynkin diagrams, we prove that the isotropy action of a pair defined by such an automorphism is equivariantly formal. The completely general case considered in Theorem 1.1 is treated in Section 5. There, we reduce the question of (equivariant) formality of an arbitrary pair to that of a pair (G,K) with G simple and conclude Theorem 1.1.

2. Preliminaries

2.1. Equivariant formality

If a compact manifold M admits a (left-)action of a compact Lie group K, the *(real) equivariant cohomology* of this action can be defined to be the cohomology of the total complex obtained from the *Cartan model*, a double complex whose degree (p,q)-term is given by

$$\left(S^p(\mathfrak{k}^*)\otimes\Omega^{q-p}(M)\right)^K .$$

Here, $S^*(\mathfrak{k}^*)$ denotes the symmetric algebra over $\mathfrak{k}^*$, whose elements we view as polynomials on the Lie algebra $\mathfrak{k}$ of K, and on which K acts by the coadjoint representation; the action of K on the space $\Omega^*(M)$ of differential forms on M is induced by pullback, and the superscript K indicates that we only consider those elements in $S^p(\mathfrak{k}^*)\otimes\Omega^{q-p}(M)$ which are invariant under the action of K induced by the respective actions on each of the factors. The action of K on M is then said to be *equivariantly formal* if the spectral sequence obtained from the filtration of the Cartan model by columns collapses at the E^1 stage, i.e. if the differential